

# HEAT KERNEL ESTIMATES FOR PSEUDODIFFERENTIAL OPERATORS, FRACTIONAL LAPLACIANS AND DIRICHLET-TO-NEUMANN OPERATORS

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**ABSTRACT.** The purpose of this article is to establish upper and lower estimates for the integral kernel of the semigroup  $\exp(-tP)$  associated to a classical, strongly elliptic pseudodifferential operator  $P$  of positive order on a closed manifold. The Poissonian bounds generalize those obtained for perturbations of fractional powers of the Laplacian. In the selfadjoint case, extensions to  $t \in \mathbb{C}_+$  are studied. In particular, our results apply to the Dirichlet-to-Neumann semigroup.

## INTRODUCTION

Let  $M$  be a compact  $n$ -dimensional Riemannian manifold and  $P$  a classical, strongly elliptic pseudodifferential operator ( $\psi$ do) on  $M$  of order  $d > 0$ . We consider upper and lower estimates for the integral kernel  $\mathcal{K}_V(x, y, t)$  of the generalized heat semigroup  $V(t) = e^{-tP}$ . Semigroups generated by such nonlocal operators have been of recent interest in different settings.

1) For a Riemannian manifold  $\widetilde{M}$  with boundary  $M$ , the Dirichlet-to-Neumann operator is a first-order pseudodifferential operator on  $M$  with principal symbol  $|\xi|$ . Arendt and Mazzeo [AM07], [AM12], initiated the study of the associated semigroup and its relation to eigenvalue inequalities, motivating later studies e.g. by Gesztesy and Mitrea [GM09] and Safarov [S08].

2) The heat kernel generated by fractional powers of the Laplacian  $\Delta^{d/2}$  and their perturbations provides another example. Sharp estimates for  $e^{-t\Delta^{d/2}}$ ,  $0 < d < 2$ , can be obtained from those for  $e^{-t\Delta}$  by subordination formulas. For perturbations on bounded domains in  $\mathbb{R}^n$ , recent work on estimates includes Chen, Kim and Song [CKS12] and other works by these authors, and Bogdan et al. [BGR10].

In this article we generalize the Poissonian estimates obtained in the second case to parameter-elliptic operators  $P$  on closed manifolds, by pseudodifferential methods. In particular, we allow nonselfadjoint operators. A main result is:

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**Theorem.** *The kernel of the semigroup satisfies*

$$(*) \quad |\mathcal{K}_V(x, y, t)| \leq C e^{-c_1 t} (d(x, y) + t^{1/d})^{-n-d}, \text{ for } x, y \in M, t \geq 0,$$

for any  $c_1$  smaller than the infimum  $\gamma(P)$  of the real part of the spectrum of  $P$ .

If  $P$  is selfadjoint  $\geq 0$ , the estimates extend to complex  $t = e^{i\theta}|t|$  for  $|\theta| < \frac{\pi}{2}$ , with uniform estimates

$$(**) \quad |\mathcal{K}_V(x, y, t)| \leq C(\cos \theta)^{-N} e^{-\gamma(P) \operatorname{Re} t} \frac{|t|}{(d(x, y) + |t|^{1/d})^d} ((d(x, y) + |t|^{1/d})^{-n} + 1),$$

where  $N = \max\{\frac{n}{d}, \frac{7n}{2} + 4d + 7\}$ .

Here  $d(x, y)$  denotes the distance between  $x$  and  $y$ . If  $P$  is a system, it suffices that  $P - \lambda$  is parameter-elliptic on the rays in a sector containing  $\{\operatorname{Re} \lambda \leq 0\}$ . Extending  $(*)$ , also derivatives of the kernel, and, if further spectral information is available, a refined description of the long-time behavior, are obtained in the paper.

For the Dirichlet-to-Neumann operator, as well as for the perturbations of fractional powers of the Laplacian of orders  $0 < d < 2$ , we get not only upper estimates but also similar lower estimates at small distances.

The estimate  $(*)$  exhibits a large class of operators which satisfy upper estimates closely related to those studied abstractly in Duong and Robinson [DR96]. As a simple application of  $(**)$  and Hölder's inequality, one can for example obtain ultracontractive estimates

$$\|e^{-tP}\|_{\mathcal{L}(L_p, L_q)} \leq C(\cos \theta)^{-N} \left\| \frac{|t|}{(d(x, y) + |t|^{1/d})^d} ((d(x, y) + |t|^{1/d})^{-n} + 1) \right\|_{L_{q,y} L_{p',x}},$$

uniformly for  $t \in \mathbb{C}_+$ . In the case of operators with Gaussian heat kernel estimates, a rich spectral theory has been developed (see e.g. Arendt [A04], Ouhabaz [O05]).

With the help of comparison principles, our result implies Poissonian estimates e.g. for boundary problems in an open subset  $\Omega$  of  $M$ : If  $P$  is the variational operator associated to a Dirichlet form  $a$  with domain  $\mathcal{D} \subset L_2(M)$ , we consider the abstract Dirichlet realization  $P_\Omega$  associated to the closure of  $a|_{\mathcal{D} \cap C_0(\Omega)}$ . In the case where  $a$  is Markovian, one obtains  $0 \leq \mathcal{K}_{e^{-tP_\Omega}} \leq \mathcal{K}_{e^{-tP}}$  on  $\Omega$ . See Grigor'yan and Hu [GH08] for more refined comparison principles.

*Outline.* Section 1 collects some known facts. In Section 2 we treat semigroups generated by nonselfadjoint  $P$  for  $t \geq 0$ . Section 3 extends the estimates to complex  $t$  for selfadjoint  $P$ . Section 4 includes lower estimates for perturbations of fractional powers of the Laplacian and for the Dirichlet-to-Neumann operator.

## 1. PRELIMINARIES

*Notation:*  $\langle \xi \rangle = \sqrt{\xi^2 + 1}$ . The indication  $\lesssim$  means “ $\leq$  a constant times”,  $\gtrsim$  means “ $\geq$  a constant times”, and  $\doteq$  means that both hold.

Let  $P$  be a classical  $\psi$ do of order  $d \in \mathbb{R}_+$ , acting in a Hermitian  $N$ -dimensional  $C^\infty$  vector bundle  $E$  over a closed, compact Riemannian  $n$ -dimensional manifold  $M$ . We assume that  $P - \lambda$  is parameter-elliptic on all rays with argument in  $]\frac{\pi}{2} - \varphi_0, \frac{3\pi}{2} + \varphi_0[$  for

some  $\varphi_0 \in ]0, \frac{\pi}{2}[$ . (In the notation of Seeley [S67], these rays are “rays of minimal growth”.) From  $P$  one can define the generalized heat operator  $V(t) = e^{-tP}$ ,  $t \geq 0$ , a holomorphic semigroup generated by  $P$ , as explained in detail e.g. in [G96], Sect. 4.2. The kernel  $\mathcal{K}_V(x, y, t)$  ( $C^\infty$  for  $t > 0$ ) was analyzed there in its dependence on  $t$ , but mainly with a view to sup-norm estimates over all  $x, y$ , allowing an analysis of the diagonal behavior, that of  $\mathcal{K}_V(x, x, t)$ . We shall expand the analysis here to give more information on  $\mathcal{K}_V(x, y, t)$ .

For convenience of the reader we recall the definitions of symbol spaces that are used. For  $d \in \mathbb{R}$ , the symbol space  $S_{1,0}^d(\mathbb{R}^n \times \mathbb{R}^n)$  consists of the  $C^\infty$ -functions  $a(x, \xi)$  ( $x, \xi \in \mathbb{R}^n$ ) such that for all  $\alpha, \beta \in \mathbb{N}_0^n$ ,

$$(1.1) \quad |D_x^\beta D_\xi^\alpha a(x, \xi)| \leq \langle \xi \rangle^{d-|\alpha|};$$

it is a Fréchet space provided with the seminorms  $\sup_{x, \xi} |\langle \xi \rangle^{-d+|\alpha|} D_x^\beta D_\xi^\alpha a|$ . The symbols define operators  $A = \text{Op}(a(x, \xi))$  of order  $d$  by

$$\text{Op}(a(x, \xi))u = \mathcal{F}^{-1}a(x, \xi)\mathcal{F}u = \int_{\mathbb{R}^n} e^{ix \cdot \xi} a(x, \xi) \hat{u}(\xi) d\xi,$$

where  $\mathcal{F}u = \hat{u}$  denotes the Fourier transform and  $d\xi = (2\pi)^{-n} d\xi$ . The operator maps from  $\mathcal{S}(\mathbb{R}^n)$  to  $\mathcal{S}(\mathbb{R}^n)$ , extending to suitable spaces of distributions and Sobolev spaces, and obeying various composition rules.

The space of *classical symbols* of order  $d$ ,  $S^d(\mathbb{R}^n \times \mathbb{R}^n)$ , is the subset of  $S_{1,0}^d(\mathbb{R}^n \times \mathbb{R}^n)$  where  $a(x, \xi)$  moreover has an asymptotic expansion  $a \sim \sum_{l \in \mathbb{N}_0} a_{d-l}$  in terms  $a_{d-l}(x, \xi)$  homogeneous in  $\xi$  of degree  $d-l$  for  $|\xi| \geq 1$ , such that  $a'_M = a - \sum_{l < M} a_{d-l} \in S_{1,0}^{d-M}$  for all  $M \in \mathbb{N}_0$ . The principal symbol  $a_d$  is often denoted  $a^0$ .

It should be noted that we here use the globally estimated symbols of Hörmander [H83], Section 18.1, which have the advantage that remainders are kept inside the calculus.

Operators on manifolds are defined by use of local coordinates and rules for change of variables, composition with cut-off functions etc.; we refer to the quoted works for details.

The book [G96] moreover includes parameter-dependent symbols  $a(x, \xi, \lambda)$  for  $\lambda$  in a sector of  $\mathbb{C}$ , with special symbol estimates involving the parameter (also operators on manifolds with boundary are treated there).

Consider a localized situation where the symbol  $p(x, \xi)$  of  $P$  is defined in a bounded open subset of  $\mathbb{R}^n$  — we can assume it is extended to  $\mathbb{R}^n$ , with symbol estimates valid uniformly in  $x$ . The abovementioned hypothesis of parameter-ellipticity means that the spectrum of the principal symbol  $p^0(x, \xi)$  (an  $N \times N$ -matrix) is contained in the sector  $\{\lambda \mid |\arg \lambda| \leq \theta_0\}$ ,  $\theta_0 = \frac{\pi}{2} - \varphi_0$ , when  $|\xi| \geq 1$ . This holds in particular when  $P$  is strongly elliptic, for then

$$(1.2) \quad \text{Re}(p^0(x, \xi)v, v) \geq c|\xi|^d|v|^2, \text{ for } |\xi| \geq 1, v \in \mathbb{C}^N, \text{ with } c > 0,$$

and hence since

$$(1.3) \quad |\text{Im}(p^0v, v)| \leq |(p^0v, v)| \leq C|\xi|^d|v|^2 \leq c^{-1}C \text{Re}(p^0v, v), \text{ for } |\xi| \geq 1, v \in \mathbb{C}^N,$$

$P$  satisfies the condition of parameter-ellipticity with  $\varphi_0 = \frac{\pi}{2} - \theta_0$ , where  $\theta_0 = \arctan(c^{-1}C) \in ]0, \frac{\pi}{2}[$ . When  $P$  is scalar, the two ellipticity properties are equivalent, but for systems,

strong ellipticity is more restrictive than the mentioned parameter-ellipticity (also called parabolicity of  $\partial_t + P$ ).

The spectrum  $\sigma(P)$  of  $P$  lies in a right half-plane and has a finite lower bound  $\gamma(P) = \inf\{\operatorname{Re} \lambda \mid \lambda \in \sigma(P)\}$ . We can modify  $p^0$  for small  $\xi$  such that  $\sigma(p^0(x, \xi))$  has a positive lower bound throughout and lies in  $\{\lambda = re^{i\theta} \mid r > 0, |\theta| \leq \theta_0\}$ .

The information in the following is taken from [G96], Section 3.3.

The resolvent  $Q_\lambda = (P - \lambda)^{-1}$  exists and is holomorphic in  $\lambda$  on a neighborhood of a set

$$(1.4) \quad W_{r_0, \varepsilon} = \{\lambda \in \mathbb{C} \mid |\lambda| \geq r_0, \arg \lambda \in [\theta_0 + \varepsilon, \pi - \theta_0 - \varepsilon], \operatorname{Re} \lambda \leq \gamma(P) - \varepsilon\}.$$

(with  $\varepsilon > 0$ ). There exists a parametrix  $Q'_\lambda$  on a neighborhood of a possibly larger set (with  $\delta > 0, \varepsilon > 0$ )

$$V_{\delta, \varepsilon} = \{\lambda \in \mathbb{C} \mid |\lambda| \geq \delta \text{ or } \arg \lambda \in [\theta_0 + \varepsilon, \pi - \theta_0 - \varepsilon]\};$$

such that this parametrix coincides with  $(P - \lambda)^{-1}$  on the intersection. Its symbol  $q(x, \xi, \lambda)$  in local coordinates is holomorphic in  $\lambda$  there and has the form

$$(1.5) \quad q(x, \xi, \lambda) \sim \sum_{l \geq 0} q_{-d-l}(x, \xi, \lambda), \text{ where } q_{-d} = (p^0(x, \xi) - \lambda)^{-1}.$$

Here when  $P$  is scalar,

$$(1.6) \quad q_{-d-1} = b_{1,1}(x, \xi)q_{-d}^2, \dots, q_{-d-l} = \sum_{k=1}^{2l} b_{l,k}(x, \xi)q_{-d}^{k+1}, \dots;$$

with symbols  $b_{l,k}$  independent of  $\lambda$  and homogeneous of degree  $dk - l$  in  $\xi$  for  $|\xi| \geq 1$ . When  $P$  is a system, each  $q_{-d-l}$  is for  $l \geq 1$  a finite sum of terms with the structure

$$(1.7) \quad r(x, \xi, \lambda) = b_1 q_{-d}^{\nu_1} b_2 q_{-d}^{\nu_2} \cdots b_M q_{-d}^{\nu_M} b_{M+1},$$

where the  $b_k$  are homogeneous  $\psi$ do symbols of order  $s_k$  independent of  $\lambda$ , the  $\nu_k$  are positive integers with sum  $\geq 2$ , and  $s_1 + \cdots + s_{M+1} - d(\nu_1 + \cdots + \nu_M) = -d - l$ . (Further information and references in Remark 3.3.7.) Moreover, the remainder  $q'_M = q - \sum_{l < M} q_{-d-l}$  satisfies for  $\lambda$  with  $|\pi - \arg \lambda| \leq \frac{\pi}{2} + \varphi$ , any  $|\varphi| < \varphi_0$ ,

$$(1.8) \quad |D_x^\beta D_\xi^\alpha q'_M(x, \xi, \lambda)| \leq \langle \xi \rangle^{d-|\alpha|-M} (1 + |\xi| + |\lambda|^{1/d})^{-2d}, \text{ when } M + |\alpha| > d.$$

(Cf. Theorems 3.3.2, and 3.3.5, applied to the rays with arguments in  $]\frac{\pi}{2} - \varphi_0, \frac{3\pi}{2} + \varphi_0[$ .)

## 2. SEMIGROUPS GENERATED BY PARAMETER-ELLIPTIC PSEUDODIFFERENTIAL OPERATORS

As explained in [G96], Section 4.2, the semigroup  $V(t) = e^{-tP}$  can be defined from  $P$  by the Cauchy integral formula

$$(2.1) \quad V(t) = \frac{i}{2\pi} \int_C e^{-t\lambda} (P - \lambda)^{-1} d\lambda,$$

where  $\mathcal{C}$  is a suitable curve going in the positive direction around the spectrum of  $P$ ; it can be taken as the boundary of  $W_{r_0, \varepsilon}$  for a small  $\varepsilon$ . In the local coordinate patch the symbol is (for any  $M \in \mathbb{N}_0$ )

$$(2.2) \quad \begin{aligned} v(x, \xi, t) &= v_{-d} + \cdots + v_{-d-M+1} + v'_M \sim \sum_{l \geq 0} v_{-d-l}(x, \xi, t), \text{ where} \\ v_{-d-l} &= \frac{i}{2\pi} \int_{\mathcal{C}} e^{-t\lambda} q_{-d-l}(x, \xi, \lambda) d\lambda, \quad v'_M = \frac{i}{2\pi} \int_{\mathcal{C}} e^{-t\lambda} q'_M d\lambda. \end{aligned}$$

A prominent example is  $e^{-t\sqrt{\Delta}}$  where  $\Delta$  denotes the (nonnegative) Laplace-Beltrami operator on  $M$ . This is a Poisson operator from  $M$  to  $M \times \overline{\mathbb{R}}_+$  as defined in the Boutet de Monvel calculus ([B71], cf. also [G96]), when  $t$  is identified with  $x_{n+1}$ . When  $M$  is replaced by  $\mathbb{R}^n$ , its kernel is the well-known Poisson kernel

$$(2.3) \quad \mathcal{K}(x, y, t) = c_n \frac{t}{|(x - y, t)|^{n+1}}$$

for the operator solving the Dirichlet problem for  $\Delta$  on  $\mathbb{R}_+^{n+1}$ .

Also more general operator families  $V(t) = e^{-tP}$  with  $P$  of order 1 are sometimes spoken of as Poisson operators (e.g. by Taylor [T81]), and indeed we can show that for  $P$  of any order  $d \in \mathbb{R}_+$ ,  $V(t)$  identifies with a Poisson operator in the Boutet de Monvel calculus. This will be accounted for in detail elsewhere. In order to match the conventions for Poisson symbol-kernels, the indexation in (2.2) is chosen slightly differently from that in [G96], Section 4.2, where  $v_{-d-l}$  would be denoted  $v_{-l}$ . We define  $V_{-d-l}(t)$  and  $V'_M(t)$  in local coordinates to be the  $\psi$ do's with symbol  $v_{-d-l}(x, \xi, t)$  resp.  $v'_M(x, \xi, t)$ . The kernel  $\mathcal{K}_V(x, y, t)$  is in local coordinates expanded according to the symbol expansion:

$$(2.4) \quad \mathcal{K}_V(x, y, t) = \sum_{0 \leq l < M} \mathcal{K}_{V_{-d-l}}(x, y, t) + \mathcal{K}_{V'_M}(x, y, t).$$

The following result follows from [G96].

**Theorem 2.1.** *1° In local coordinates, the kernel terms satisfy for some  $c' > 0$ :*

$$(2.5) \quad |\mathcal{K}_{V_{-d-l}}(x, y, t)| \leq e^{-c't} \begin{cases} t^{(l-n)/d} & \text{if } d-l > -n, \\ t(|\log t| + 1) & \text{if } d-l = -n, \\ t & \text{if } d-l < -n. \end{cases}$$

*For a given  $c_0 > 0$  we can modify  $p^0$  to satisfy  $\inf_{x, \xi} \gamma(p^0(x, \xi)) \geq c_0$ ; then  $c'$  can be any number in  $]0, c_0[$ .*

*2° Moreover, with the modification in 1° used with  $c_0 = \gamma(P)$  if  $\gamma(P) > 0$ , the remainder satisfies*

$$(2.6) \quad |\mathcal{K}_{V'_M}(x, y, t)| \leq e^{-c_1 t} \begin{cases} t^{(M-n)/d} & \text{if } d-M > -n, \\ t(|\log t| + 1) & \text{if } d-M = -n, \\ t & \text{if } d-M < -n, \end{cases}$$

for any  $c_1 < \gamma(P)$ . In particular,

$$(2.7) \quad |\mathcal{K}_V(x, y, t)| \leq e^{-c_1 t} t^{-n/d}.$$

*Proof.* The theorem was shown with slightly less precision on the constants  $c', c_1$  in [G96], Theorems 4.2.2 and 4.2.5. It was there aimed towards applications where  $d$  is integer. The estimates of resolvent symbols in Section 3.3 are still valid when  $d \in \mathbb{R}_+$ , but the replacement of  $P$  by  $P + a$  ( $a \in \mathbb{R}$ ) in the beginning of Section 4.2 on heat operators only gives a classical  $\psi$ do when  $d$  is integer, so we need another device to take the value of  $\gamma(P)$  into account for general  $d \in \mathbb{R}_+$ . We shall now explain the needed modifications, with reference to [G96].

For  $1^\circ$ , the proof in Theorem 4.2.2 shows the validity of (2.5) with a small positive  $c' < \inf_{x, \xi} \gamma(p^0(x, \xi))$ . For a given  $c_0 > 0$ , the proof goes through to allow any  $c' < c_0$ , when  $p^0(x, \xi)$  is modified for  $|\xi| \leq R$  (for a possibly large  $R$ ) to satisfy  $\inf \gamma(p^0(x, \xi)) \geq c_0$ .

For  $2^\circ$ , the remainder symbol  $q'_M$  is holomorphic on  $W_{r_0, \varepsilon}$ ; here if  $\gamma(P) > 0$  we define the terms  $q_{-d-l}$  as under  $1^\circ$ , with  $c_0 = \gamma(P)$ . For large  $M$ ,  $q'_M$  is  $\leq \langle \lambda \rangle^{-2}$ . The proof of Th. 4.2.2 gives an estimate of  $\mathcal{K}_{V'_M}$  by  $e^{-c_1 t} t(1 + |\log t|)$ , and the proof of Theorem 4.2.5 shows how to remove the logarithm. The estimates of  $\mathcal{K}_{V'_M}$  for lower values of  $M$  follow by addition of the estimates of finitely many  $\mathcal{K}_{V_{-d-l}}$ -terms.  $\square$

We shall improve this to give information on the dependence on  $|x - y|$  also. This will rely on the following result on kernels of  $S_{1,0}^r$ - $\psi$ do's, found e.g. in Taylor [T81], Lemma XII 3.1, or [T96], Proposition VII 2.2.

**Proposition 2.2.** *Let  $a \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$  be such that for some  $r \in \mathbb{R}$ ,  $N \in \mathbb{N}_0$  with  $N > n + r$ , and all  $0 \leq |\alpha| \leq N$ ,*

$$(2.8) \quad \sup_{x, \xi} \langle \xi \rangle^{-r+|\alpha|} |D_\xi^\alpha a(x, \xi)| < \infty.$$

*Then the inverse Fourier transform  $\mathcal{K}_A(x, y) = \mathcal{F}_{\xi \rightarrow z}^{-1} a(x, \xi)|_{z=x-y}$  is  $O(|x - y|^{-N})$  for  $|x - y| \rightarrow \infty$ , and satisfies for  $|x - y| > 0$ :*

$$(2.9) \quad |\mathcal{K}_A(x, y)| \leq \begin{cases} |x - y|^{-r-n} & \text{if } r > -n, \\ |\log |x - y|| + 1 & \text{if } r = -n, \\ 1 & \text{if } r < -n. \end{cases}$$

*In particular, if  $a \in S_{1,0}^r(\mathbb{R}^n \times \mathbb{R}^n)$  defining the  $\psi$ do  $A$ , the estimates hold for its kernel  $\mathcal{K}_A(x, y)$  for all  $N > n + r$ , each estimate depending only on the listed symbol seminorms.*

The dependence on  $N$  follows from an inspection of the proof.

In the scalar case the kernel study can be based on nice explicit formulas, that we think are worth explaining. Consider the contribution from one of the terms in (1.6). As integration curve we can here use  $C_\theta$  consisting of the two rays  $re^{i\theta}$  and  $re^{-i\theta}$ ,  $\theta = \theta_0 + \varepsilon$ . For  $t > 0$ , a replacement of  $t\lambda$  by  $\varrho$  gives:

$$(2.10) \quad \begin{aligned} w_{l,k}(x, \xi, t) &= \frac{i}{2\pi} \int_{C_\theta} e^{-t\lambda} \frac{b_{l,k}(x, \xi)}{(p^0(x, \xi) - \lambda)^{k+1}} d\lambda = \frac{i}{2\pi} \int_{C_\theta} e^{-\varrho} \frac{t^k b_{l,k}}{(tp^0 - \varrho)^{k+1}} d\varrho \\ &= \frac{i}{2\pi} t^k b_{l,k} \int_{C_{\theta,R}} \frac{e^{-\varrho}}{(tp^0 - \varrho)^{k+1}} d\varrho = \frac{1}{k!} t^k b_{l,k} e^{-tp^0}; \end{aligned}$$

here we have replaced the integration curve by a closed curve  $C_{\theta,R}$  connecting the two rays by a circular piece in the right half-plane with radius  $R \geq 2t|p^0(x, \xi)|$ , and applied the Cauchy integral formula for derivatives of holomorphic functions. This shows:

$$(2.11) \quad v_{-d} = e^{-tp^0}, \quad v_{-d-l}(x, \xi, t) = \sum_{k=1}^{2l} \frac{1}{k!} t^k b_{l,k}(x, \xi) e^{-tp^0(x, \xi)} \text{ for } l \geq 1.$$

Then the kernels of the  $V_{-d-l}(t)$  can be estimated by the following observations.

**Proposition 2.3.** *Let  $p^0(x, \xi)$  be the principal symbol of a classical scalar strongly elliptic pseudo  $P$  on  $\mathbb{R}^n$  of order  $d \in \mathbb{R}_+$ , chosen such that  $\operatorname{Re} p^0(x, \xi) \geq c_0 > 0$ .*

*1° Let  $c' \in [0, c_0[$ . For any  $j \in \mathbb{N}_0$ ,  $(t(p^0(x, \xi) - c'))^j e^{-t(p^0(x, \xi) - c')}$  is in  $S_{1,0}^0(\mathbb{R}^n \times \mathbb{R}^n)$  uniformly in  $t \geq 0$ .*

*2° Let*

$$(2.12) \quad w(x, \xi, t) = \frac{i}{2\pi} \int_{C_\theta} e^{-t\lambda} \frac{b(x, \xi)}{(p^0(x, \xi) - \lambda)^{k+1}} d\lambda,$$

where  $k \geq 1$  and  $b \in S_{1,0}^{dk-l}(\mathbb{R}^n \times \mathbb{R}^n)$ . Then

$$(2.13) \quad w(x, \xi, t) = \frac{1}{k!} t^k b(x, \xi) e^{-tp^0(x, \xi)} = e^{-c't} t w'(x, \xi, t),$$

where  $w'(x, \xi, t) \in S_{1,0}^{d-l}(\mathbb{R}^n \times \mathbb{R}^n)$ , uniformly for  $t \geq 0$ .

Moreover,  $\tilde{w}(x, z, t) = \mathcal{F}_{\xi \rightarrow z}^{-1} w$  satisfies for any  $c' \in ]0, c_0[$ :

$$(2.14) \quad |\tilde{w}(x, z, t)| \leq e^{-c't} \begin{cases} t |z|^{l-d-n} & \text{if } d-l > -n, \\ t (|\log |z|| + 1) & \text{if } d-l = -n, \\ t & \text{if } d-l < -n. \end{cases}$$

It follows that for  $l \geq 1$ ,  $\mathcal{K}_{V_{-d-l}}(x, y, t) = \mathcal{F}_{\xi \rightarrow z}^{-1} v_{-d-l}(x, \xi, t)|_{z=x-y}$  satisfies the estimates

$$(2.15) \quad |\mathcal{K}_{V_{-d-l}}(x, y, t)| \leq e^{-c't} \begin{cases} t |x-y|^{l-d-n} & \text{if } d-l > -n, \\ t (|\log |x-y|| + 1) & \text{if } d-l = -n, \\ t & \text{if } d-l < -n. \end{cases}$$

*Proof.* 1°. For each fixed  $t > 0$ ,  $e^{-tp^0(x, \xi)}$  is rapidly decreasing in  $\xi$ , hence is in  $S_{1,0}^{-\infty}$ . But for our purposes we need estimates that hold uniformly in  $t$  for  $t \rightarrow 0$ . Let

$$M_{j,k,l} = \sup_{s \geq 0} s^l \partial_s^k (s^j e^{-s}).$$

Then for  $t \geq 0$ ,  $\xi \in \mathbb{R}^n$ ,

$$(2.16) \quad \begin{aligned} & |(tp^0(x, \xi))^j e^{-tp^0(x, \xi)}| \leq M_{j,0,0}, \\ & |\partial_{\xi_i} ((tp^0)^j e^{-tp^0})| = |\partial_s (s^j e^{-s})|_{s=tp^0} t \partial_{\xi_i} p^0| \leq M_{j,k,1} |(p^0)^{-1} \partial_{\xi_i} p^0| \leq \langle \xi \rangle^{-1}, \dots \\ & |\partial_{\xi}^\alpha ((tp^0)^j e^{-tp^0})| \leq \langle \xi \rangle^{-|\alpha|}, \dots \end{aligned}$$

showing the assertion for  $c' = 0$ . (2.16) holds also if  $p^0$  is replaced by  $p^0 - c'$  throughout, when  $c' \in ]0, c_0[$ .

2°. The first identity in (2.13) was shown in (2.10). We can also write

$$w(x, \xi, t) = \frac{1}{k!} t b(p^0 - c')^{1-k} (t(p^0 - c'))^{k-1} e^{-c't} e^{-t(p^0 - c')} = e^{-c't} t w'(x, \xi, t).$$

Here  $b(p^0 - c')^{1-k}$  is in  $S_{1,0}^{d-l}$ , independent of  $t$ , and by 1°,  $(t(p^0 - c'))^{k-1} e^{-t(p^0 - c')}$  is uniformly in  $S_{1,0}^0$ , so it follows that  $w'$  is uniformly in  $S_{1,0}^{d-l}$ . We can now apply Proposition 2.2 to draw the conclusion (2.14).

Since  $v_{-d-l}(x, \xi, t)$  is a sum of such terms when  $l \geq 1$ , the estimates (2.15) follow.  $\square$

For systems  $P$  we can use systematic estimates from [G96]. We find for general  $P$ :

**Theorem 2.4.** 1° *In local coordinates,  $\mathcal{K}_{V_{-d}}$  satisfies for some  $c' > 0$ :*

$$(2.17) \quad |\mathcal{K}_{V_{-d}}(x, y, t)| \stackrel{\cdot}{\leq} e^{-c't} t |x - y|^{-d-n}.$$

*For  $l \geq 1$ , the kernels  $\mathcal{K}_{V_{-d-l}}$  satisfy (2.15). If  $\gamma(P) > 0$ , we modify  $p^0$  to satisfy  $\inf_{x, \xi} \gamma(p^0(x, \xi)) \geq \gamma(P)$ , then  $c'$  can be any number in  $]0, \gamma(P)[$ .*

2° *Moreover, with  $p^0$  chosen as in 1°,*

$$(2.18) \quad |\mathcal{K}_{V_M'}(x, y, t)| \stackrel{\cdot}{\leq} e^{-c_1 t} \begin{cases} t |x - y|^{M-d-n} & \text{if } d - M > -n, \\ t (|\log |x - y|| + 1) & \text{if } d - M = -n, \\ t & \text{if } d - M < -n, \end{cases}$$

*for any  $c_1 < \gamma(P)$ . In particular,*

$$(2.19) \quad |\mathcal{K}_V(x, y, t)| \stackrel{\cdot}{\leq} e^{-c_1 t} t |x - y|^{-d-n}.$$

*Proof.* 1°. When  $P$  is scalar, the estimates in (2.15) for  $l \geq 1$  are shown in Proposition 2.3, when we take  $c_0 = \gamma(P)$  if  $\gamma(P) > 0$ . For general systems  $P$ , the symbols  $q_{-d-l}$  are sums of symbols as in (1.7), and we apply [G96], Lemma 4.2.3. Here (4.2.35) with  $k = -d - l$  shows that

$$|D_x^\beta D_\xi^\alpha v_{-d-l}(x, \xi, t)| \stackrel{\cdot}{\leq} \langle \xi \rangle^{d-l-|\alpha|} t e^{-c't},$$

for all  $\alpha, \beta$ . Actually, the estimate (4.2.35) has  $e^{-ct\langle \xi \rangle^d}$  with a positive  $c$  as the last factor, but an inspection of the proof (the location of integral contours) shows that  $e^{-ct\langle \xi \rangle^d}$  can be replaced by  $e^{-c't}$ , if  $c' < \inf \gamma(p^0(x, \xi))$ . This shows that  $e^{c't} t^{-1} v_{-d-l}$  is in  $S_{1,0}^{d-l}$  uniformly in  $t$ , so the estimates of the  $\mathcal{K}_{V_{-d-l}}$  follow by use of Proposition 2.2.

For  $l = 0$ , we can argue as follows in the scalar case: For each  $j = 1, \dots, n$ ,

$$\partial_{\xi_j} v_{-d} = \partial_{\xi_j} e^{-tp^0} = -t(\partial_{\xi_j} p^0) e^{-tp^0},$$

where  $\partial_{\xi_j} p^0 \in S_{1,0}^{d-1}$ . Now as in Proposition 2.3,  $e^{-c't} \partial_{\xi_j} p^0 e^{-t(p^0 - c')}$  is in  $S_{1,0}^{d-1}$  uniformly in  $t$ , and hence  $\tilde{v}_{-d} = \mathcal{F}_{\xi \rightarrow z}^{-1} v_{-d}$  satisfies, since  $d - 1 > -n$ ,

$$(2.20) \quad |z_j \tilde{v}_{-d}| \stackrel{\cdot}{\leq} e^{-c't} t |z|^{-d+1-n}.$$



Taking the square root of the sum of squares for  $j = 1, \dots, n$ , we find after division by  $|z|$  that

$$(2.21) \quad |\tilde{v}_{-d}| \leq e^{-c't} t |z|^{-d-n}.$$

In the systems case we note that

$$(2.22) \quad \partial_{\xi_j} q_{-d} = -q_{-d} (\partial_{\xi_j} p^0) q_{-d},$$

since  $\partial_{\xi_j} [(p^0 - \lambda)(p^0 - \lambda)^{-1}] = 0$ . Lemma 4.2.3 applies to this in the same way as above, showing that

$$|D_x^\beta D_\xi^\alpha \partial_{\xi_j} v_{-d}(x, \xi, t)| \leq \langle \xi \rangle^{d-1-|\alpha|} t e^{-c't},$$

so  $e^{c't} t \partial_{\xi_j} v_{-d}$  is uniformly in  $S_{1,0}^{d-1}$ . We conclude (2.20), from which (2.21) follows, implying (2.17).

2°. Here the estimate in (2.18) has already been shown for large  $M$  in Theorem 2.1. For lower values of  $M$ , we can add the estimates of the entering homogeneous terms  $\mathcal{K}_{V_{-d-l}}$  with  $l \geq M$ ; the top term gives the weakest estimate. (It is used that  $x$  and  $y$  need only run in a bounded set, for the contribution from the localized piece.)  $\square$

Theorems 2.1 and 2.4 together lead to Poisson-like kernel estimates:

**Theorem 2.5.** 1° *One has in local coordinates:*

$$(2.23) \quad |\mathcal{K}_{V_{-d-l}}(x, y, t)| \leq e^{-c't} \begin{cases} t (|x - y| + t^{1/d})^{l-d-n} & \text{if } d - l > -n, \\ t (|\log(|x - y| + t^{1/d})| + 1) & \text{if } d - l = -n, \\ t & \text{if } d - l < -n, \end{cases}$$

for some  $c' > 0$ . If  $\gamma(P) > 0$ , we modify  $p^0$  to satisfy  $\inf_{x,\xi} \gamma(p^0(x, \xi)) \geq \gamma(P)$ ; then  $c'$  can be any number in  $]0, \gamma(P)[$ .

2° Moreover, with  $p^0$  chosen as in 1°,

$$(2.24) \quad |\mathcal{K}_{V'_M}(x, y, t)| \leq e^{-c_1 t} \begin{cases} t (|x - y| + t^{1/d})^{M-d-n} & \text{if } d - M > -n, \\ t (|\log(|x - y| + t^{1/d})| + 1) & \text{if } d - M = -n, \\ t & \text{if } d - M < -n, \end{cases}$$

for any  $c_1 < \gamma(P)$ . In particular,

$$(2.25) \quad \begin{aligned} |\mathcal{K}_V(x, y, t)| &\leq e^{-c_1 t} t (|x - y| + t^{1/d})^{-d-n}, \\ |\mathcal{K}_{V'_1}(x, y, t)| &\leq e^{-c_1 t} t (|x - y| + t^{1/d})^{1-d-n}. \end{aligned}$$

3° For the operators defined on  $M$ , one has (with  $d(x, y)$  denoting the distance between  $x$  and  $y$ )

$$(2.26) \quad |\mathcal{K}_V(x, y, t)| \leq e^{-c_1 t} t (d(x, y) + t^{1/d})^{-d-n},$$

for any  $c_1 < \gamma(P)$ .

*Proof.* 1°–2°. In the region where  $|x - y| \geq t^{1/d}$ ,

$$|x - y| \leq |x - y| + t^{1/d} \leq 2|x - y|,$$

in other words,  $|x - y| \doteq |x - y| + t^{1/d}$ . Then the estimates in Theorem 2.4 imply the validity of the above estimates on this region.

In the region where  $|x - y| \leq t^{1/d}$ , we have instead that  $t^{1/d} \doteq |x - y| + t^{1/d}$ . Then the estimates in Theorem 2.1 imply the above estimates on that region; for example

$$t^{-n/d} = t(t^{1/d})^{-d-n} \doteq t(|x - y| + t^{1/d})^{-d-n}$$

there. For the two regions together, this shows (2.23)–(2.25).

3°. This follows from the estimates in local coordinates.  $\square$

When the eigenvalues of  $P$  with real part equal to  $\gamma(P)$  (necessarily finitely many) are semisimple (i.e., the algebraic multiplicity equals the geometric multiplicity), we can sharpen the information on the behavior for  $t \rightarrow \infty$ :

**Corollary 2.6.** *Assume that all eigenvalues of  $P$  with real part  $\gamma(P)$  are semisimple (it holds in particular when  $P$  is selfadjoint). Then*

$$(2.27) \quad |\mathcal{K}_{e^{-tP}}(x, y, t)| \dot{\leq} e^{-\gamma(P)t} \frac{t}{(d(x, y) + t^{1/d})^d} \left( (d(x, y) + t^{1/d})^{-n} + 1 \right).$$

*Proof.* The spectral projections  $\Pi_j = \frac{i}{2\pi} \int_{\mathcal{C}_j} (P - \lambda)^{-1} d\lambda$  onto the eigenspaces  $X_j$  for the eigenvalues  $\{\lambda_1, \dots, \lambda_k\}$  with real part  $\gamma(P)$  (where  $\mathcal{C}_j$  is a small circle around the eigenvalue), are pseudodifferential operators of order  $-\infty$ , and their kernels  $\mathcal{K}_{\Pi_j}(x, y)$  are bounded. If  $\varepsilon > 0$ , the operator  $P' = P + \varepsilon \sum_{j=1}^k \Pi_j$  satisfies  $\gamma(P') > \gamma(P)$ . By Theorem 2.5 applied to  $P'$ ,

$$|\mathcal{K}_{e^{-tP'}}(x, y, t)| \dot{\leq} e^{-\gamma(P)t} t (d(x, y) + t^{1/d})^{-d-n}.$$

On the other hand,  $V(t) = e^{-tP'} + (1 - e^{-\varepsilon t}) \sum_{j=1}^k e^{-t\lambda_j} \Pi_j$ , so

$$\mathcal{K}_{e^{-tP}}(x, y, t) = \mathcal{K}_{e^{-tP'}}(x, y, t) + (1 - e^{-\varepsilon t}) \sum_{j=1}^k e^{-t\lambda_j} \mathcal{K}_{\Pi_j}(x, y).$$

From

$$1 - e^{-\varepsilon t} \leq \min\{1, \varepsilon t\} \dot{\leq} \frac{t}{(\text{diam}(M) + t^{1/d})^d} \leq \frac{t}{(d(x, y) + t^{1/d})^d},$$

we conclude that  $(1 - e^{-\varepsilon t})|\mathcal{K}_{\Pi_j}(x, y)| \dot{\leq} \frac{t}{(d(x, y) + t^{1/d})^d}$ , and (2.27) follows since  $|e^{-t\lambda_j}| = e^{-t\gamma(P)}$  for each  $j$ .  $\square$

**Remark 2.7.** The proof of Corollary 2.6 allows to sharpen the estimates in Theorem 2.5 and Theorem 2.9 below also in the general case where the eigenvalues with real part  $\gamma(P)$  are not all semisimple. Denote by  $r$  the dimension of the largest irreducible  $P$ -invariant subspace of any eigenspace  $X_j$  associated to an eigenvalue with real part  $\gamma(P)$ . Then in Theorems 2.5 and 2.9 we may replace the upper bound  $e^{-c't}t(d(x, y) + t^{1/d})^{-d-n-k}$  by

$$(2.28) \quad e^{-\gamma(P)t}(1 + t^{r-1}) \frac{t}{(d(x, y) + t^{1/d})^d} ((d(x, y) + t^{1/d})^{-n-k} + 1).$$

It is not hard to extend the estimates to complex  $t$  in a sector around  $\mathbb{R}_+$ . Namely, since  $p^0$  has its spectrum in the sector  $\{|\arg \lambda| \leq \theta_0\}$ ,  $e^{i\varphi}P$  satisfies the parameter-ellipticity condition when  $|\varphi| < \varphi_0 = \frac{\pi}{2} - \theta_0$ . For each  $\varphi$  it generates a semigroup  $e^{-te^{i\varphi}P}$ , and these operator families coincide with the holomorphic extension of  $V(t)$  to the rays  $\{re^{i\varphi}\}$  in the sector  $V_{\varphi_0} = \{t \in \mathbb{C} \mid |\arg t| < \varphi_0\}$ . On each ray we have the estimates in Theorem 2.5, they hold uniformly in closed subsectors of  $V_{\varphi_0}$ . We have hereby obtained:

**Theorem 2.8.** *With  $\varphi_0$  and  $\theta_0$  defined as in the beginning of Section 1, the semigroup generated by  $P$  extends holomorphically to the sector  $\{|\arg t| < \varphi_0\}$ , and the estimates in Theorem 2.5 hold in terms of  $|t|$  on any closed sector  $\{|\arg t| \leq \varphi\}$  with  $0 < \varphi < \varphi_0$ , taking  $c_1 < \min_{|\varphi| \leq \varphi} \gamma(e^{i\varphi}P)$ .*

More information in the case where  $P$  is selfadjoint will be given in Section 3 below.

Also the derivatives of the kernels can be estimated by use of the symbol estimates in [G96].

**Theorem 2.9.** *1° One has in local coordinates:*

$$(2.29) \quad |D_x^\beta D_y^\gamma D_t^j \mathcal{K}_{V_{-d-l}}(x, y, t)| \leq e^{-c't} \begin{cases} t(|x - y| + t^{1/d})^{l-(1+j)d-|\gamma|-n} & \text{if } (j+1)d + |\gamma| - l > -n, \\ t(|\log(|x - y| + t^{1/d})| + 1) & \text{if } (j+1)d + |\gamma| - l = -n, \\ t & \text{if } (j+1)d + |\gamma| - l < -n, \end{cases}$$

for some  $c' > 0$ . If  $\gamma(P) > 0$ , we modify  $p^0$  to satisfy  $\inf_{x, \xi} \gamma(p^0(x, \xi)) \geq \gamma(P)$ ; then  $c'$  can be any number in  $]0, \gamma(P)[$ .

2° Moreover, with  $p^0$  chosen as in 1°,

$$(2.30) \quad |D_x^\beta D_y^\gamma D_t^j \mathcal{K}_{V'_M}(x, y, t)| \leq e^{-c_1 t} \begin{cases} t(|x - y| + t^{1/d})^{M-(j+1)d-|\gamma|-n} & \text{if } (j+1)d + |\gamma| - M > -n, \\ t(|\log(|x - y| + t^{1/d})| + 1) & \text{if } (j+1)d + |\gamma| - M = -n, \\ t & \text{if } (j+1)d + |\gamma| - M < -n, \end{cases}$$

for any  $c_1 < \gamma(P)$ .

3° The estimates of derivatives of  $\mathcal{K}_V$  hold for the operator defined on  $M$  with  $|x - y|$  replaced by  $d(x, y)$ .

*Proof.* As in Theorem 2.7, the estimates are pieced together from estimates generalizing those in Theorem 2.1 resp. Theorem 2.4 to include derivatives. We use that

$$\begin{aligned} |D_x^\beta D_y^\gamma D_t^j \mathcal{K}_{V_{-d-l}}(x, y, t)| &= |D_x^\beta D_z^\gamma D_t^j \tilde{v}_{-d-l}(x, z, t)|_{z=x-y} \\ &= |\mathcal{F}_{\xi \rightarrow z}^{-1}(\xi^\gamma D_x^\beta D_t^j v_{-d-l}(x, \xi, t))|_{z=x-y}. \end{aligned}$$

To generalize Theorem 2.1 to allow  $x$ - and  $y$ -derivatives we just have to apply the arguments of [G96] Theorems 4.2.2 and 4.2.5 to the modified symbols  $\xi^\gamma D_x^\beta v_{-d-l}$ , to get the estimates (2.29) with  $|x - y|$  replaced by 0. Derivatives with respect to  $t$  alone are explained in Theorem 4.2.5; finally this is combined with  $x$ - and  $y$ -derivatives in a straightforward way. Similar considerations work for remainders; here we can in fact refer directly to (4.2.60) for large  $M$ , and the statements for lower  $M$  follow by addition of the appropriate set of estimates of  $\mathcal{K}_{V_{-d-l}}$ -terms. This gives the expected generalization of Theorem 2.1, namely (2.29)–(2.30) with  $|x - y|$  replaced by 0.

For the generalization of Theorem 2.4 we note that estimates

$$|\xi^\gamma D_x^\beta D_\xi^\alpha D_t^j v_{-d-l}(x, \xi, t)| \leq \langle \xi \rangle^{(j+1)d + |\gamma| - |\alpha| - l} t e^{-c't}$$

for  $|\alpha| + l > 0$ , all  $\beta, j$ , follow from [G96] Lemma 4.2.3 (see the remarks around (4.2.40) for how to include  $t$ -derivatives, as done also in Theorem 4.2.5). Thus  $e^{c't} t^{-1} \xi^\gamma D_x^\beta D_t^j v_{-d-l}$  is in  $S_{1,0}^{(j+1)d + |\gamma| - l}$  uniformly in  $t$ , and it follows by Proposition 2.2 that

$$|D_z^\gamma D_x^\beta D_t^j \tilde{v}_{-d-l}(x, z, t)| \leq e^{-c't} \begin{cases} t |z|^{-(j+1)d - |\gamma| + l - n}, & \text{if } (j+1)d + |\gamma| - l > -n, \\ t (|\log(|z| + t^{1/d})| + 1) & \text{if } (j+1)d + |\gamma| - l = -n, \\ t & \text{if } (j+1)d + |\gamma| - l < -n. \end{cases}$$

This implies estimates as in (2.29) with  $|x - y| + t^{1/d}$  replaced by  $|x - y|$ . The conclusion is immediate for  $l \geq 1$ , and for  $l = 0$ , we use the estimates of  $D_{\xi_j} v$  as in the proof of Theorem 2.4. Again for remainder estimates, we can appeal to (4.2.60) for large  $M$ .

The proof is now completed as in Theorem 2.7.  $\square$

### 3. ESTIMATES IN THE COMPLEX PLANE FOR SELFADJOINT OPERATORS

In this section we shall derive some uniform estimates for the extension of the semigroup into the region  $\mathbb{C}_+ = \{t \in \mathbb{C} \mid \operatorname{Re} t > 0\}$ , when  $P$  is selfadjoint. To do so, we need to account for how the estimates of symbols and remainders like (1.6–8) depend on  $\arg \lambda$ . We assume for simplicity that  $P \geq 0$ .

As already observed in Theorem 2.7,  $V(t)$  exists for  $t \in \mathbb{C}_+$ . There are uniform estimates on closed subsectors, but to describe the behavior for rays near the imaginary axis we need estimates of  $q(x, \xi, \lambda)$  for  $\lambda$  near  $\mathbb{R}_+$ .

As in [G96], we denote  $|\lambda|^{1/d} = \mu$ , and write  $\langle (\xi, \mu) \rangle = (1 + |\xi|^2 + \mu^2)^{1/2}$  for short as  $\langle \xi, \mu \rangle$ ; it is  $\doteq (1 + |\xi| + |\lambda|^{1/d})$ .

**Proposition 3.1.** *Let  $P$  be selfadjoint  $\geq 0$  and let  $\lambda \in \mathbb{C}$  with  $\arg \lambda = \varphi \in ]0, \frac{\pi}{2}[$ . Then*

$$(3.1) \quad \begin{aligned} |q_{-d}(x, \xi, \lambda)| &= |(p^0(x, \xi) - \lambda)^{-1}| \leq (\sin \varphi)^{-1} \langle \xi, \mu \rangle^{-d}, \\ |D_x^\beta D_\xi^\alpha q_{-d}(x, \xi, \lambda)| &\leq (\sin \varphi)^{-1 - |\alpha| - |\beta|} \langle \xi \rangle^{d-l-|\alpha|} \langle \xi, \mu \rangle^{-2d}, \text{ when } |\alpha| + |\beta| > 0. \end{aligned}$$

For all  $l, \alpha, \beta$  with  $l > 0$ ,

$$(3.2) \quad |D_x^\beta D_\xi^\alpha q_{-d-l}(x, \xi, \lambda)| \leq (\sin \varphi)^{-2l-|\alpha|-|\beta|} \langle \xi \rangle^{d-l-|\alpha|} \langle \xi, \mu \rangle^{-2d}.$$

*Proof.* We have for  $\lambda = e^{i\varphi}|\lambda|$  with  $\varphi \in ]0, \frac{\pi}{2}[$ ,  $v \in \mathbb{C}^N$ , since  $p^0(x, \xi)$  is symmetric with lower bound  $\geq c\langle \xi \rangle^d$ :

$$\begin{aligned} |(p^0 v, v) - \lambda |v|^2| &\geq |\operatorname{Im}((p^0 v, v) - |\lambda| e^{i\varphi} |v|^2)| = |\lambda| \sin \varphi |v|^2, \\ |(p^0 v, v) - \lambda |v|^2| &= |e^{-i\varphi}(p^0 v, v) - |\lambda| |v|^2| \geq |\operatorname{Im} e^{-i\varphi}(p^0 v, v)| \\ &= \sin \varphi (p^0 v, v) \geq \sin \varphi c \langle \xi \rangle^d |v|^2, \end{aligned}$$

from which follows

$$|(p^0 - \lambda)v||v| \geq |((p^0 - \lambda)v, v)| \geq \sin \varphi (|\lambda| + \langle \xi \rangle^d) |v|^2.$$

This implies that  $|(p^0 - \lambda)^{-1}| \leq (\sin \varphi)^{-1} (\langle \xi \rangle^d + |\lambda|)^{-1} \doteq (\sin \varphi)^{-1} \langle \xi, \mu \rangle^{-d}$ , showing (3.1).

The other estimates follow as in [G96] from the structure of the terms in the parametrix, using (3.1):  $q_{-d-l}$  is for  $l \geq 1$  a finite sum of terms, where  $\nu_1 + \dots + \nu_M \geq 2$  takes values up to  $2l$ ,

$$r(x, \xi, \lambda) = b_1 q_{-d}^{\nu_1} b_2 q_{-d}^{\nu_2} \dots b_M q_{-d}^{\nu_M} b_{M+1},$$

cf. (1.7). Each  $q_{-d}$  contributes with a factor  $(\sin \varphi)^{-1}$ , and there are up to  $2l$  such factors; this shows (3.2) for  $\alpha = \beta = 0$ . Each differentiation may hit a factor  $q_{-d}$  giving an extra  $(\sin \varphi)^{-1}$  in view of (2.22); this leads to the estimates (3.2) by the Leibniz formula.  $\square$

Estimates of remainders  $Q'_M$  and their symbols  $q'_M$  are more difficult to work out, since they depend on the interplay between the exact resolvent  $Q_\lambda$  and the homogeneous symbol terms, and they will be more costly in powers of  $(\sin \varphi)^{-1}$  the larger  $M$  is taken. We shall here go directly to remainder *kernel* estimates.

More precisely, we consider the kernel of the operator  $Q'_M = Q_\lambda - \sum_{l < M} Q_{-d-l}$ , where each  $Q_{-d-l}$  is an operator on the manifold  $M$  constructed from the symbols  $q_{-d-l}$  in local coordinates.

For  $\lambda$  as in Lemma 3.1,

$$\|(P - \lambda)u\| \|u\| \geq |((P - \lambda)u, u)| \geq \operatorname{Im} \lambda \|u\|^2 = \sin \varphi |\lambda| \|u\|^2;$$

hence the resolvent  $Q_\lambda = (P - \lambda)^{-1}$  has operator norm  $\leq (\sin \varphi |\lambda|)^{-1}$  in  $L_2(M)$ . The operator  $Q'_M = Q_\lambda - \sum_{l < M} Q_{-d-l}$  is a  $\psi$ do of order  $-d - M$  (for each  $\lambda$ ). Moreover,

$$(3.3) \quad \begin{aligned} Q'_M &= Q'_M (P - \lambda) Q_\lambda = \tilde{R}_M Q_\lambda, \text{ where} \\ \tilde{R}_M &= Q'_M (P - \lambda) = 1 - \sum_{l < M} Q_{-d-l} (P - \lambda) \end{aligned}$$

is a  $\psi$ do of order  $-M$ . Now

$$\|\tilde{R}_M Q_\lambda\|_{\mathcal{L}(L_2, H^s)} \leq (\sin \varphi |\lambda|)^{-1} \|\tilde{R}_M\|_{\mathcal{L}(L_2, H^s)}.$$

For  $s > n$ , it is known that a  $\psi$ do continuous from  $L_2(M)$  to  $H^s(M)$  has a continuous kernel estimated by the operator norm; then the kernel of  $Q'_M = \tilde{R}_M Q_\lambda$  is continuous and is estimated by

$$(3.4) \quad |\mathcal{K}_{Q'_M}(x, y, \lambda)| \leq \|Q'_M\|_{\mathcal{L}(L_2, H^s)} \leq (\sin \varphi |\lambda|)^{-1} \|\tilde{R}_M\|_{\mathcal{L}(L_2, H^s)}.$$

In preparation for the study of  $\tilde{R}_M$  and its dependence on  $\lambda$  we prove a lemma on a typical composition formula, treated by basic methods in the  $\psi$ do theory.

**Lemma 3.2.** *Let  $b(x, \xi) \in S_{1,0}^{d_2}(\mathbb{R}^n \times \mathbb{R}^n)$ , and let  $a(x, \xi, \lambda) \in S_{1,0}^{d_1}(\mathbb{R}^n \times \mathbb{R}^n)$  with respect to  $(x, \xi)$ , with  $\lambda$  as in Proposition 3.1, such that for some  $d' \geq 0$ ,  $N \in \mathbb{R}$  one has for all  $\alpha, \beta \in \mathbb{N}_0^n$ ,*

$$(3.5) \quad |D_x^\beta D_\xi^\alpha a(x, \xi, \lambda)| \leq (\sin \varphi)^{-N-|\alpha|-|\beta|} \langle \xi \rangle^{d'+d_1-|\alpha|} \langle \xi, \mu \rangle^{-d'}.$$

1° *There exists  $c(x, \xi, \lambda) \in S_{1,0}^{d_1+d_2}(\mathbb{R}^n \times \mathbb{R}^n)$  such that  $\text{Op}(a) \text{Op}(b) = \text{Op}(c)$ , and for every  $M \in \mathbb{N}_0$ ,*

$$(3.6) \quad c(x, \xi, \lambda) = \sum_{|\alpha| < M} \frac{1}{\alpha!} D_\xi^\alpha a(x, \xi, \lambda) \partial_x^\alpha b(x, \xi) + r_M(a, b),$$

where

$$(3.7) \quad |D_x^\beta D_\xi^\alpha r_M(a, b)| \leq (\sin \varphi)^{-N-M-|\alpha|-|\beta|} \langle \xi \rangle^{d'+d_1+d_2-M-|\alpha|} \langle \xi, \mu \rangle^{-d'}.$$

2° *If (3.5) for  $\alpha = \beta = 0$  is replaced by*

$$(3.8) \quad |a(x, \xi, \lambda)| \leq (\sin \varphi)^{-N} \langle \xi \rangle^{d+d_1} \langle \xi, \mu \rangle^{-d}.$$

for some  $0 \leq d \leq d'$ , then (3.6) holds with (3.7) valid for  $M \geq 1$  and the estimates of  $r_0$  replaced by

$$(3.9) \quad |D_x^\beta D_\xi^\alpha r_0(a, b)| \leq (\sin \varphi)^{-N-|\alpha|-|\beta|} \langle \xi \rangle^{d+d_1+d_2-|\alpha|} \langle \xi, \mu \rangle^{-d}.$$

3° *For  $\gamma \in \mathbb{N}_0^n$ ,  $D^\gamma \text{Op}(a) = \text{Op}(a^\gamma)$ , where*

$$(3.10) \quad |D_x^\beta D_\xi^\alpha a^\gamma(x, \xi, \lambda)| \leq \sum_{k \leq |\gamma|} (\sin \varphi)^{-N-k-|\alpha|-|\beta|} \langle \xi \rangle^{d'+d_1+|\gamma|-k-|\alpha|} \langle \xi, \mu \rangle^{-d'}.$$

*Proof.* 1°. Let  $\chi(x, \xi)$  denote a  $C^\infty$ -function that is 1 for  $|x|^2 + |\xi|^2 \leq 1$  and vanishes for  $|x|^2 + |\xi|^2 \geq 2$ , then we can replace the given symbols by their products with  $\chi(\varepsilon x, \varepsilon \xi)$ , which makes all integrals calculated below convergent. It is known in the theory (by the technique of oscillatory integrals, cf. [H83], Sect. 7.8), that the resulting symbols converge to the given symbols for  $\varepsilon \rightarrow 0$  in all the seminorms that are involved. The modified symbols will again be denoted  $a, b$ . We can also assume that  $b$  has compact support in  $x$  (in a set containing the  $x$  for which we need the formula). Then  $\hat{b}(\eta, \xi) = \mathcal{F}_{x \rightarrow \eta} b(x, \xi)$  satisfies

$$(3.11) \quad |D_\xi^\alpha \hat{b}(\eta, \xi)| \leq \langle \eta \rangle^{-N'} \langle \xi \rangle^{d_2-|\alpha|},$$

for all  $\alpha, N'$ . It follows from the  $\psi$ do defining formula that  $\text{Op}(a) \text{Op}(b) = \text{Op}(c)$ , where

$$(3.12) \quad \begin{aligned} c(x, \xi, \lambda) &= \int_{\mathbb{R}^{4n}} a(x, \eta, \lambda) b(y, \xi) e^{i(x-y) \cdot \eta} e^{i(y-z) \cdot \xi} dz d\xi dy d\eta \\ &= \int_{\mathbb{R}^n} a(x, \xi + \eta, \lambda) \hat{b}(\eta, \xi) e^{ix \cdot \eta} d\eta. \end{aligned}$$

If  $M > 0$ , we insert the Taylor expansion of  $a$  in  $\xi$  up to order  $M$ ,

$$\begin{aligned} a(x, \xi + \eta, \lambda) &= \sum_{|\alpha| < M} \frac{1}{\alpha!} \eta^\alpha \partial_\xi^\alpha a(x, \xi, \lambda) \\ &\quad + \sum_{|\alpha| = M} \frac{M}{\alpha!} \eta^\alpha \int_0^1 (1-h)^{M-1} \partial_\xi^\alpha a(x, \xi + h\eta, \lambda) dh, \end{aligned}$$

obtaining that  $c = c_{<M} + r_M$ , where

$$\begin{aligned} c_{<M} &= (2\pi)^{-n} \int_{\mathbb{R}^n} \sum_{|\alpha| < M} \frac{1}{\alpha!} \partial_\xi^\alpha a(x, \xi, \lambda) \eta^\alpha \hat{b}(\eta, \xi) e^{ix \cdot \eta} d\eta \\ &= \sum_{|\alpha| < M} \frac{1}{\alpha!} \partial_\xi^\alpha a(x, \xi, \lambda) D_x^\alpha b(x, \xi) = \sum_{|\alpha| < M} \frac{1}{\alpha!} D_\xi^\alpha a(x, \xi, \lambda) \partial_x^\alpha b(x, \xi), \\ r_M &= (2\pi)^{-n} \int_{\mathbb{R}^n} \sum_{|\alpha| = M} \frac{M}{\alpha!} \int_0^1 (1-h)^{M-1} \partial_\xi^\alpha a(x, \xi + h\eta, \lambda) dh \eta^\alpha \hat{b}(\eta, \xi) e^{ix \cdot \eta} d\eta. \end{aligned}$$

The sum over  $|\alpha| < M$  equals the sum in (3.6). For the last integral we use that

$$\begin{aligned} |\partial_\xi^\alpha a(x, \xi + h\eta, \lambda)| &\leq (\sin \varphi)^{-N-M} \langle \xi + h\eta \rangle^{d'+d_1-M} \langle \xi + h\eta, \mu \rangle^{-d'} \\ &\leq (\sin \varphi)^{-N-M} \langle \xi \rangle^{d'+d_1-M} \langle \xi, \mu \rangle^{-d'} \langle \eta \rangle^{|d'+d_1-M|+d'}, \end{aligned}$$

by the Peetre inequality. Taking this together with the estimates (3.11) of  $\hat{b}$  (with a large  $N'$ ), we can conclude that

$$|r_M| \leq (\sin \varphi)^{-N-M} \langle \xi \rangle^{d'+d_1+d_2-M} \langle \xi, \mu \rangle^{-d'}.$$

For  $M = 0$ , we apply such considerations directly to  $r_0 = c(x, \xi, \lambda)$  in (3.12):

$$|r_0| \leq (\sin \varphi)^{-N} \int \langle \xi + \eta \rangle^{d'+d_1} \langle \xi + \eta, \mu \rangle^{-d'} \langle \eta \rangle^{-N'} \langle \xi \rangle^{d_2} d\eta \leq (\sin \varphi)^{-N} \langle \xi \rangle^{d'+d_1+d_2} \langle \xi, \mu \rangle^{-d'}.$$

Derivatives of  $r_M$  in  $x$  and  $\xi$  are treated in a similar way.

In the case 2° the proof goes through in a similar way, except that  $d'$  is replaced by  $d$  in expressions containing undifferentiated factors  $a$ .

In 3°, the  $\lambda$ -independent factor is to the left, and (3.12) holds with integrand  $(\xi + \eta)^\gamma \mathcal{F}_{z \rightarrow \eta} a(z, \xi, \lambda) e^{ix \cdot \eta}$ . The Taylor expansion of  $(\xi + \eta)^\gamma$  is a finite binomial expansion  $\sum_{\kappa \leq \gamma} \binom{\gamma}{\kappa} \xi^{\gamma-\kappa} \eta^\kappa$  and leads to a finite composition formula where the estimates (3.10) of the terms can be read off directly.  $\square$

The composed symbol  $c = r_0(a, b)$  is also denoted  $a \circ b$  (used in [G96]) or  $a \# b$ .

The lemma will be used in the following investigation of the symbol of  $\tilde{R}_M$ . We denote  $P - \lambda = \tilde{P}$ , with the parameter-dependent symbol  $\tilde{p}(x, \xi, \lambda) = p(x, \xi) - \lambda$  in local coordinates; here for any  $M \in \mathbb{N}_0$ ,

$$\begin{aligned} (3.13) \quad P &= \sum_{k < M} p_{d-k} + p'_M, \quad \tilde{P} = \sum_{k < M} \tilde{p}_{d-k} + \tilde{p}'_M, \text{ with} \\ \tilde{p}_d &= p - \lambda, \quad \tilde{p}_{d-k} = p_{d-k} \text{ for } k > 0, \quad \tilde{p}'_M = p'_M \text{ for } M > 0. \end{aligned}$$

$p_d$  is also denoted  $p^0$ . The  $p_{d-k}$  are homogeneous in  $|\xi|$  of degree  $d - k$  for  $|\xi| \geq 1$ , and  $p'_M \in S_{1,0}^{d-M}(\mathbb{R}^n \times \mathbb{R}^n)$ .

**Proposition 3.3.** *Let  $M \geq 1$ . The symbol  $\tilde{r}_M(x, \xi, \lambda)$  of  $\tilde{R}_M$  (cf. (3.3)) satisfies in local coordinates:*

$$(3.14) \quad |D_x^\beta D_\xi^\alpha \tilde{r}_M(x, \xi, \lambda)| \leq (\sin \varphi)^{-2M+1-|\alpha|-|\beta|} \langle \xi \rangle^{d-M-|\alpha|} \langle \xi, \mu \rangle^{-d}.$$

*Proof.* We have that

$$\tilde{r}_M = 1 - \sum_{k < M} \sum_{l < M} q_{-d-l} \circ \tilde{p}_{d-k} - \sum_{l < M} q_{-d-l} \circ \tilde{p}'_M.$$

The terms in the parametrix symbol  $\sum_{l \geq 0} q_{-d-l}$  are constructed as solutions to the successive equations for  $m \in \mathbb{N}_0$ :

$$(3.15) \quad \sum_{|\alpha|+k+l=m} \frac{1}{\alpha!} D_\xi^\alpha q_{-d-l} \partial_x^\alpha \tilde{p}_{d-k} = \begin{cases} 1 & \text{for } m = 0, \\ 0 & \text{for } m = 1, 2, \dots, \end{cases}$$

cf. e.g. Seeley [S67], (1). We can use the truncated composition formula in Lemma 3.2 to compute the symbol  $\tilde{r}_M$  of  $\tilde{R}_M$  with expansions in up to  $M$  homogeneous terms:

$$\begin{aligned} \tilde{r}_M = 1 - \sum_{k < M} \sum_{l < M} & \left\{ \sum_{k+l+|\alpha| < M} \frac{1}{\alpha!} D_\xi^\alpha q_{-d-l} \partial_x^\alpha \tilde{p}_{d-k} + r_{M-k-l}(q_{-d-l}, \tilde{p}_{d-k}) \right\} \\ & - r_0 \left( \sum_{l < M} q_{-d-l}, \tilde{p}'_M \right). \end{aligned}$$

By (3.15),

$$\sum_{k < M} \sum_{l < M} \sum_{k+l+|\alpha| < M} \frac{1}{\alpha!} D_\xi^\alpha q_{-d-l} \partial_x^\alpha \tilde{p}_{d-k} = 1.$$

Thus  $\tilde{r}_M$  consists of the following terms:

$$\tilde{r}_M = - \sum_{k < M} \sum_{l < M} r_{M-k-l}(q_{-d-l}, \tilde{p}_{d-k}) - r_0 \left( \sum_{l < M} q_{-d-l}, \tilde{p}'_M \right).$$

Using the estimates (3.1) and (3.2) together with  $|D_x^\beta D_\xi^\alpha \tilde{p}_{d-k}(x, \xi)| \leq \langle \xi \rangle^{d-k-|\alpha|}$ , we obtain from Lemma 3.2 with  $d' = 2d$ ,  $d_1 = -d-l$  and  $d_2 = d-k$  that for  $l \geq 1$ :

$$(3.16) \quad \begin{aligned} & |D_x^\beta D_\xi^\alpha r_{M-k-l}(q_{-d-l}, \tilde{p}_{d-k})| \\ & \leq (\sin \varphi)^{-M+k+l-2l-|\alpha|-|\beta|} \langle \xi \rangle^{2d-d-l+d-k-(M-k-l)-|\alpha|} \langle \xi, \mu \rangle^{-2d} \\ & \leq (\sin \varphi)^{-2M+1-|\alpha|-|\beta|} \langle \xi \rangle^{2d-M-|\alpha|} \langle \xi, \mu \rangle^{-2d}, \end{aligned}$$

since  $k-l \geq -M+1$ .

For  $l = 0$  we find in view of Lemma 3.2 2°, since  $k < M$ ,

$$(3.17) \quad \begin{aligned} & |D_x^\beta D_\xi^\alpha r_{M-k}(q_{-d}, \tilde{p}_{d-k})| \\ & \leq (\sin \varphi)^{-1-(M-k)-|\alpha|-|\beta|} \langle \xi \rangle^{2d-d+d-k-(M-k)-|\alpha|} \langle \xi, \mu \rangle^{-2d} \\ & \leq (\sin \varphi)^{-M-1-|\alpha|-|\beta|} \langle \xi \rangle^{2d-M-|\alpha|} \langle \xi, \mu \rangle^{-2d}. \end{aligned}$$



Finally,

$$\begin{aligned}
 (3.18) \quad & |D_x^\beta D_\xi^\alpha r_0 \left( \sum_{l < M} q_{-d-l}, \tilde{p}'_M \right)| \\
 & \leq (\sin \varphi)^{-1-|\alpha|-|\beta|} \langle \xi \rangle^{d-M-|\alpha|} \langle \xi, \mu \rangle^{-d} \\
 & \quad + \sum_{l < M} (\sin \varphi)^{-2l-|\alpha|-|\beta|} \langle \xi \rangle^{2d-d-l+(d-M)-|\alpha|} \langle \xi, \mu \rangle^{-2d} \\
 & \leq (\sin \varphi)^{-2M+1-|\alpha|-|\beta|} \langle \xi \rangle^{d-M-|\alpha|} \langle \xi, \mu \rangle^{-d}.
 \end{aligned}$$

An addition of the contributions (using that  $\langle \xi \rangle / \langle \xi, \mu \rangle \leq 1$ ) gives (3.14).  $\square$

Now let us show how the kernel of  $V'_M(t)$  is estimated. Similarly to Corollary 2.6, it will be convenient to write  $P = P^\varepsilon - \varepsilon \Pi_0$ , where  $\Pi_0$  is the orthogonal projection onto the zero eigenspace of  $P$ , and  $\varepsilon > 0$  is chosen  $\leq$  the lowest positive eigenvalue, whereby  $P^\varepsilon = P + \varepsilon \Pi_0$  is  $\geq \varepsilon$ . Here  $\Pi_0$  is the  $\psi$ do of order 0 with kernel  $\sum_{j=1}^\nu \varphi_j(x) \varphi_j(y)^*$ , for an orthonormal basis  $\varphi_1, \dots, \varphi_\nu$  of the zero eigenspace. Then

$$V(t) = V^\varepsilon(t) + (1 - e^{-\varepsilon t}) \Pi_0, \text{ where } V^\varepsilon(t) = e^{-tP^\varepsilon};$$

and it is the latter operator that needs investigation.  $V^\varepsilon(t)$  is defined from the resolvent  $Q_\lambda^\varepsilon = Q_\lambda - (\varepsilon - \lambda)^{-1} \Pi_0 = (P^\varepsilon - \lambda)^{-1}$  by

$$(3.19) \quad V^\varepsilon(t) = \frac{i}{2\pi} \int_{\mathcal{C}} e^{-t\lambda} Q_\lambda^\varepsilon d\lambda.$$

When  $t \in \mathbb{C}_+$  with argument  $\arg t = \theta \in [0, \frac{\pi}{2}[$ , say, we must assure that  $\operatorname{Re} \lambda t \rightarrow -\infty$  on the integral curve  $\mathcal{C}$ . With  $\frac{\pi}{2} - \theta$  denoted  $2\varphi$ , this is assured if  $\lambda$  runs on a contour formed of the rays  $\lambda = r e^{\pm i\varphi}$ , connected near 0 by a circle of radius  $\varepsilon' < \varepsilon$  passing *to the right of* 0. Denote  $\inf_{\lambda \in \mathcal{C}} \operatorname{Re} \lambda = c_1 > 0$ .

For simplicity of notation we drop the  $\varepsilon$ -index in the next calculations.

**Remark 3.4.** The argument  $\theta = \frac{\pi}{2} - 2\varphi$  of  $t$  runs in  $[0, \frac{\pi}{2}[$ , when  $\varphi$  runs in  $]0, \frac{\pi}{4}]$ . On this interval,  $\sin \varphi \doteq \sin(2\varphi) = \cos \theta$ , so they can be used interchangeably in our estimates.

To find  $V'_M(t)$  we can plug  $Q'_M$  into the integral (3.19) and study the kernels, trying to get an estimate in terms of  $|t|$ . This cannot quite be achieved with (3.19), but better estimates are obtained if we first make use of the resolvent formula

$$Q_\lambda = -\lambda^{-1} + \lambda^{-1} Q_\lambda P.$$

This gives

$$\begin{aligned}
 (3.20) \quad & V(t) = \frac{i}{2\pi} \int_{\mathcal{C}} e^{-t\lambda} (-\lambda^{-1} + \lambda^{-1} Q_\lambda P) d\lambda = \frac{i}{2\pi} \int_{\mathcal{C}} e^{-t\lambda} \lambda^{-1} Q_\lambda P d\lambda, \\
 & \partial_t V(t) = -\frac{i}{2\pi} \int_{\mathcal{C}} e^{-t\lambda} Q_\lambda P d\lambda.
 \end{aligned}$$

Thus  $\partial_t \mathcal{K}_{V'_M}$  is the kernel of the integral of the  $M$ -th remainder  $-(Q_\lambda P)'_M$  of  $-Q_\lambda P$ . We know that  $\mathcal{K}_{V'_M}$  vanishes at  $t = 0$ , and want to show boundedness of the last integral applied to the kernel of the  $M$ -th remainder. Here (cf. also (3.3))

$$Q_\lambda P = \left( \sum_{l < M} Q_{-d-l} + Q'_M \right) P = \sum_{l < M} Q_{-d-l} P + \tilde{R}_M Q_\lambda P = \sum_{l < M} Q_{-d-l} P + \tilde{R}_M P Q_\lambda.$$

The term  $\tilde{R}_M P Q_\lambda$  already belongs to the  $M$ -th remainder  $(Q_\lambda P)'_M$ , and its kernel is estimated by

$$\|\tilde{R}_M P\|_{\mathcal{L}(L_2, H^{n+1})} (\sin \varphi |\lambda|)^{-1}$$

where  $\tilde{R}_M P$  can be treated by another application of Lemma 3.2.

We shall here need an estimate of  $L_2$ -bounds in terms of symbol seminorms. Many variants are known, and we use the following, found in Marschall [M87], Theorem 2.1.

**Theorem 3.5.** *Let  $a \in S_{1,0}^0(\mathbb{R}^n \times \mathbb{R}^n)$  be such that for some  $C > 0$ ,  $N \in \mathbb{N}_0$  with  $N > \frac{n}{2}$ , and all  $\alpha, \beta \in \mathbb{N}_0^n$  with  $0 \leq |\alpha| \leq N$ ,  $0 \leq |\beta| \leq 1$ ,*

$$(3.21) \quad \sup_{x,\xi} \langle \xi \rangle^{-|\alpha|} |D_x^\beta D_\xi^\alpha a(x, \xi)| \leq C < \infty.$$

*Then the associated operator  $A = \text{Op}(a)$  is bounded on  $L_2(\mathbb{R}^n)$ , and  $\|A\|_{\mathcal{L}(L_2(\mathbb{R}^n))} \leq C$ .*

The dependence of the operator norm on  $C$  follows from an inspection of the proof.

**Proposition 3.6.** *The symbol  $\tilde{r} = r_0(\tilde{r}_M, p)$  of  $\tilde{R} = \tilde{R}_M P$  satisfies*

$$(3.22) \quad |D_x^\beta D_\xi^\alpha \tilde{r}| \leq (\sin \varphi)^{-2M+1-|\alpha|-|\beta|} \langle \xi \rangle^{2d-M-|\alpha|} \langle \xi, \mu \rangle^{-d}.$$

Moreover, for  $M \geq n + 1 + 2d$ ,

$$(3.23) \quad \begin{aligned} \|\tilde{R}\|_{\mathcal{L}(L_2, H^{n+1})} &\leq (\sin \varphi)^{-2M-3n/2-2} \langle \lambda \rangle^{-1}, \text{ and hence} \\ \|\tilde{R}_M P Q_\lambda\|_{\mathcal{L}(L_2, H^{n+1})} &\leq (\sin \varphi)^{-2M-3n/2-3} \langle \lambda \rangle^{-1} |\lambda|^{-1}. \end{aligned}$$

*Proof.* An application of Lemma 3.2 1° to the composition  $\tilde{R} = \tilde{R}_M P$  shows (3.22). Next,

$$\|\tilde{R}\|_{\mathcal{L}(L_2, H^{n+1})} = \left( \sum_{|\gamma| \leq n+1} \|D^\gamma \circ \tilde{R}\|_{\mathcal{L}(L_2)}^2 \right)^{\frac{1}{2}} = \left( \sum_{|\gamma| \leq n+1} \|\tilde{R}^\gamma\|_{\mathcal{L}(L_2)}^2 \right)^{\frac{1}{2}},$$

where an application of Lemma 3.2 3° shows that for  $|\gamma| \leq n + 1$ ,

$$|D_x^\beta D_\xi^\alpha \tilde{r}^\gamma(x, \xi, \lambda)| \leq (\sin \varphi)^{-2M+1-(n+1)-|\beta|-|\alpha|} \langle \xi \rangle^{2d-M+(n+1)-|\alpha|} \langle \lambda \rangle^{-1}.$$

According to Theorem 3.5,  $\tilde{R}^\gamma$  is bounded in  $L_2$  when  $M \geq 2d + n + 1$ , with operator norm estimated by the seminorms of the symbol derivatives with  $|\alpha| \leq N$ ,  $|\beta| \leq 1$  for some  $N > n/2$ . Hence the norm is estimated by

$$(\sin \varphi)^{-2M-n-1-(n/2+\delta)} \langle \lambda \rangle^{-1} \leq (\sin \varphi)^{-2M-3n/2-2} \langle \lambda \rangle^{-1},$$

where  $\delta$  equals  $\frac{1}{2}$  or 1. Summing over  $|\gamma| \leq n + 1$ , we obtain the first line in (3.23), and the second follows, since  $\|Q_\lambda\|_{\mathcal{L}(L_2)} \leq (\sin \varphi |\lambda|)^{-1}$ .  $\square$

For the term  $\sum_{l < M} Q_{-d-l} P$  we can apply Lemma 3.2 to the compositions, much as in Proposition 3.3.

**Proposition 3.7.** *For the  $M$ -th remainder  $\tilde{R}'_M = (\sum_{l < M} Q_{-d-l} P)'_M$  of  $\sum_{l < M} Q_{-d-l} P$ , the symbol of  $\sum_{l < M} Q_{-d-l} P$  is in local coordinates a sum of homogeneous terms of degree  $0, -1, \dots, -M+1$  plus a remainder  $\tilde{r}'_M$  of the form*

$$(3.24) \quad \tilde{r}'_M = q_{-d} p'_M + \tilde{r}_M,$$

where

$$(3.25) \quad |D_x^\beta D_\xi^\alpha \tilde{r}_M(x, \xi, \lambda)| \leq (\sin \varphi)^{-2M+1-|\alpha|-|\beta|} \langle \xi \rangle^{2d-M-|\alpha|} \langle \xi, \mu \rangle^{-2d},$$

*Proof.* We shall use estimates from the proof of Proposition Y. Recall from (3.13) that  $\tilde{p}$  only differs from  $p$  in the principal term. We have

$$q_{-d-l} \circ p_{k-d} = \sum_{k+l+|\alpha| < M} \frac{1}{\alpha!} D_\xi^\alpha q_{-d-l} \partial_x^\alpha p_{d-k} + r_{M-k-l}(q_{-d-l}, p_{d-k}),$$

where the terms in the sum are homogeneous of degrees  $> -M$ , and the remainders are estimated as in (3.16) for  $l > 0$ , (3.17) for  $l = 0$ , contributing to (3.25). It remains to consider

$$(3.26) \quad \sum_{l < M} q_{-d-l} \circ p'_M = \sum_{l < M} r_0(q_{-d-l}, p'_M).$$

For  $l > 0$ ,  $r_0(q_{-d-l}, p'_M)$  satisfies estimates as in (3.25). For  $l = 0$ ,

$$r_0(q_{-d}, p'_M) = q_{-d} p'_M + r_1(q_{-d}, p'_M),$$

where  $r_1(q_{-d}, p'_M)$  satisfies estimates as in (3.25) by Lemma 3.2 2°. We collect the terms in

$$\tilde{r}_M = \sum_{k+l < M} r_{M-k-l}(q_{-d-l}, p_{d-k}) + \sum_{0 < l < M} r_0(q_{-d-l}, p'_M) + r_1(q_{-d}, p'_M),$$

then (3.24) holds with (3.25).  $\square$

Then we can finally show the estimate of the remainder kernel:

**Theorem 3.8.** *Let  $P$  be selfadjoint strongly elliptic of order  $d > 0$  on  $M$ , with  $\gamma(P) \geq 0$ . The remainder kernel  $\mathcal{K}_{V'_M}$  satisfies for  $M \geq n+1+2d$ ,  $t \in \mathbb{C}_+$  with  $\arg t = \frac{\pi}{2} - 2\varphi \in [0, \frac{\pi}{2}[$ ,*

$$(3.26) \quad |\mathcal{K}_{V'_M}(x, y, t)| \leq (\sin \varphi)^{-2M-3n/2-3} e^{-c' \operatorname{Re} t} \min\{|t|, 1\},$$

where  $c' > 0$  if  $\gamma(P) > 0$ ,  $c' = 0$  if  $\gamma(P) = 0$ .

*Proof.* If  $\gamma(P) > 0$  we use the preceding estimates directly to analyse

$$\partial_t \mathcal{K}_{V'_M} = -\frac{i}{2\pi} \int_{\mathcal{C}} e^{-t\lambda} \mathcal{K}_{(Q_\lambda P)'_M} d\lambda.$$

Here  $(Q_\lambda P)'_M = \tilde{R}_M P Q_\lambda + \tilde{R}'_M$ , cf. Propositions Z and V. They contribute as follows:

From  $\tilde{R}_M PQ_\lambda$  we get using (3.23)  
(3.28)

$$\begin{aligned} & \left| \frac{i}{2\pi} \int_{\mathcal{C}} e^{-t\lambda} \mathcal{K}_{(\tilde{R}_M PQ_\lambda)'_M} d\lambda \right| \\ & \leq (\sin \varphi)^{-2M-3n/2-3} \left( \int_{\varepsilon}^{\infty} e^{-\sin \varphi |t|r} \langle r \rangle^{-1} r^{-1} dr + \int_{|\varphi'| \leq \varphi} e^{-c_1 \operatorname{Re} t} \langle \varepsilon \rangle^{-1} |\varepsilon|^{-1} d\varphi' \right) \\ & \leq (\sin \varphi)^{-2M-3n/2-3} e^{-c' \operatorname{Re} t}, \end{aligned}$$

for some  $c' > 0$ .

From  $\tilde{R}'_M$  we get two terms. One is the operator with symbol  $\tilde{r}_M$ , which satisfies the estimates (3.25). Since the kernel equals  $\mathcal{F}_{\xi \rightarrow z}^{-1} \tilde{r}_M|_{z=x-y}$ , we find for  $M \geq n+1+2d$  that

$$|\mathcal{K}_{\tilde{R}_M}(x, y, \lambda)| \leq (\sin \varphi)^{-2M+1} \langle \lambda \rangle^{-2}.$$

Insertion in the integral as in (3.28) gives a bound  $(\sin \varphi)^{-2M+1} e^{-c' \operatorname{Re} t}$ .

To find the contribution from  $q_{-d} p'_M$ , we first perform the integration:

$$\frac{i}{2\pi} \int_{\mathcal{C}} e^{-t\lambda} (p^0 - \lambda)^{-1} p'_M d\lambda = e^{-tp^0} p'_M.$$

Here we can borrow an estimate from Lemma 3.9 below. By (3.33),

$$\begin{aligned} |D_x^\beta D_\xi^\alpha (e^{-tp^0} p'_M)| & \leq (\sin \varphi)^{-M-|\alpha|-|\beta|} \langle \xi \rangle^{-M-|\alpha|} e^{-c \operatorname{Re} t} \langle \xi \rangle^d \\ & \leq (\sin \varphi)^{-M-|\alpha|-|\beta|} \langle \xi \rangle^{-M-|\alpha|} e^{-c' \operatorname{Re} t}, \text{ hence} \\ |\mathcal{K}_{\operatorname{Op}(e^{-p^0 t} p'_M)}| & \leq (\sin \varphi)^{-M} e^{-c' \operatorname{Re} t} \text{ when } M \geq n+1. \end{aligned}$$

Since the latter estimates are dominated by that from  $\tilde{R}_M PQ_\lambda$ , we conclude that  $|\partial_t \mathcal{K}_{V'_M}(x, y, t)|$  is estimated as in (3.28). Then an integration with respect to  $t$  using that  $\mathcal{K}_{V'_M}(x, y, 0) = 0$  shows (3.26).

In the case where  $\gamma(P) = 0$ , the above considerations will be valid for  $V^\varepsilon(t)$  as in (3.19). We then have to add  $(1 - e^{-\varepsilon t}) \Pi_0$ , which has a smooth kernel bounded by  $\min\{|t|, 1\}$  and we reach the conclusion in the theorem.  $\square$

Now we turn to the contribution from the homogeneous terms.

**Lemma 3.9.** *Let  $M \in \mathbb{N}$ , let  $\sigma_1, \dots, \sigma_M$  be nonnegative integers with*

$$(3.29) \quad \sigma = \sigma_1 + \dots + \sigma_M \geq 1,$$

*and let  $f(x, \xi, \lambda)$  be a (matrix-formed) symbol of the form*

$$(3.30) \quad f(x, \xi, \lambda) = f_1(p_d - \lambda)^{-\sigma_1} f_2(p_d - \lambda)^{-\sigma_2} \dots (p_d - \lambda)^{-\sigma_M} f_{M+1},$$

*where the  $f_j(x, \xi)$  are  $\psi$ do symbols of order  $s_j \in \mathbb{R}$ , homogeneous for  $|\xi| \geq 1$ . Denote  $s_1 + \dots + s_{M+1} = s$ , then the order of  $f$  is  $k = s - \sigma d$ . Let  $F_\lambda = \operatorname{Op}(f(x, \xi, \lambda))$  on  $\mathbb{R}^n$ , and let  $E(t)$  be the operator family defined from  $F_\lambda$  for  $\operatorname{Re} t > 0$  by*

$$(3.31) \quad E(t) = \frac{i}{2\pi} \int_{\mathcal{C}} e^{-t\lambda} F_\lambda d\lambda,$$

when  $t = e^{i(\frac{\pi}{2}-2\varphi)}|t|$ . Then  $E(t) = \text{Op}(e(x, t, \xi))$ , where the symbol

$$(3.32) \quad e(x, t, \xi) = \frac{i}{2\pi} \int_{\mathcal{C}} e^{-t\lambda} f(x, \xi, \lambda) d\lambda$$

satisfies:

$$(3.33) \quad \begin{aligned} \text{(i)} \quad & e(x, s^{-d}t, s\xi) = s^{d+k} e(x, t, \xi) \text{ for } |\xi| \geq 1, s \geq 1, \\ \text{(ii)} \quad & |D_x^\beta D_\xi^\alpha e(x, t, \xi)| \leq (\sin \varphi)^{-\sigma-|\alpha|-|\beta|} \langle \xi \rangle^{d+k-|\alpha|} e^{-c \operatorname{Re} t \langle \xi \rangle^d}. \end{aligned}$$

The kernel of  $E(t)$  satisfies for  $d+k > -n$

$$(3.34) \quad |\mathcal{K}_E(x, y, t)| \leq (\sin \varphi)^{-\sigma-(d+k+n)/d} e^{-c' \operatorname{Re} t} |t|^{-(d+k+n)/d},$$

with  $c' > 0$ .

If  $\sigma \geq 2$ ,

$$(3.35) \quad |D_x^\beta D_\xi^\alpha e(x, t, \xi)| \leq (\sin \varphi)^{-\sigma-|\alpha|-|\beta|} |t| \langle \xi \rangle^{2d+k-|\alpha|} e^{-c' \operatorname{Re} t \langle \xi \rangle^d}.$$

In this case, the kernel satisfies for  $d+k \leq -n$

$$(3.36) \quad |\mathcal{K}_E(x, y, t)| \leq (\sin \varphi)^{-\sigma} e^{-c' \operatorname{Re} t} \begin{cases} |t| (|\log \operatorname{Re} t| + 1) & \text{if } d+k = -n, \\ |t| & \text{if } d+k < -n. \end{cases}$$

*Proof.* As in [G96], Lemma 4.2.3, we can pass the operator definition through the integral. To estimate  $e$ , we first consider  $|\xi| \leq 1$ . We use the residue theorem and that  $p^0$  is selfadjoint to obtain

$$|e(t, x, \xi)| = \left| \frac{i}{2\pi} \int_{\mathcal{C}} e^{-t\lambda} f(x, \xi, \lambda) d\lambda \right| \leq (1 + |t|^{\sigma-1}) e^{-c \operatorname{Re} t}.$$

Here,  $c = \gamma(p^0(x, \xi))$ .

For  $|\xi| \geq 1$ , we replace  $\mathcal{C}$  by a closed, homogeneous curve  $\mathcal{C}_{c,C}$  around the spectrum of  $p^0(x, \xi)$ .  $\mathcal{C}_{c,C}$  coincides with  $\mathcal{C}$  on an annulus of inner radius  $c|\xi|^d$  and outer radius  $C|\xi|^d$ , is closed by the segments of the boundary of this annulus which lie to the right of  $\mathcal{C}$ . Then by homogeneity

$$|e(t, x, \xi)| = \left| \frac{i}{2\pi} \int_{\mathcal{C}_{c,C}} e^{-t\lambda} f(x, \xi, \lambda) d\lambda \right| \leq (\sin \varphi)^{-\sigma} \langle \xi \rangle^d \langle \xi \rangle^k e^{-\frac{c}{2} \operatorname{Re} t |\xi|^d}.$$

Combining the two estimates, we conclude (3.33) for  $\alpha = \beta = 0$ . The derivatives  $D_x^\beta D_\xi^\alpha e(x, t, \xi)$  are of a similar form, with  $k$  and  $\sigma$  replaced by  $k - |\alpha|$  resp.  $\sigma + |\alpha| + |\beta|$ .

To show (3.34) for  $d+k > -n$ , we estimate  $\mathcal{K}_E$  by comparing  $e$  with its homogeneous extension  $e^h$ :

$$\mathcal{K}_E(x, y, t) = \int_{\mathbb{R}^n} e^{i(x-y) \cdot \xi} e^h(x, t, \xi) d\xi + \int_{|\xi| \leq 1} e^{i(x-y) \cdot \xi} (e - e^h) d\xi.$$

Using (3.33) and a homogeneous variant,

$$|\mathcal{K}_E(x, y, t)| \leq (\sin \varphi)^{-\sigma} e^{-c_1 \operatorname{Re} t} \int_{\mathbb{R}^n} e^{-c_2 \operatorname{Re} t |\xi|^d} |\xi|^{d+k} d\xi \\ + (\sin \varphi)^{-\sigma} e^{-c_1 \operatorname{Re} t} \int_{|\xi| \leq 1} e^{-c_2 \operatorname{Re} t |\xi|^d} (\langle \xi \rangle^{d+k} + |\xi|^{d+k}) d\xi .$$

The first integral is  $\doteq (\operatorname{Re} t)^{-(d+k+n)/d} \doteq (\sin \varphi)^{-(d+k+n)/d} |t|^{-(d+k+n)/d}$ , while the second remains bounded as  $|t| \rightarrow 0$ .

Now consider the case where  $\sigma \geq 2$ . As  $|f| \leq \langle \lambda \rangle^{-2}$  away from  $\mathbb{R}_+$ , the integral converges uniformly in  $t \geq 0$ . We may deform  $\mathcal{C}$  to a closed curve in the left half-plane, where  $f$  is holomorphic, to conclude  $e(x, 0, \xi) = 0$ . Also, using  $(-\lambda)(p_d - \lambda)^{-1} = 1 - p_d(p_d - \lambda)^{-1}$ ,

$$\partial_t e(x, t, \xi) = \frac{i}{2\pi} \int_{\mathcal{C}} e^{-t\lambda} (-\lambda) f(x, \xi, \lambda) d\lambda$$

can be expressed in terms of  $e$  and a second term of the same form, with one of the  $s_j$  replaced by  $s_j + d$ . By (3.33)

$$|\partial_t e(x, t, \xi)| \leq (\sin \varphi)^{-\sigma} \langle \xi \rangle^{2d+k} e^{-c \operatorname{Re} t \langle \xi \rangle^d}$$

and hence, since the value at  $t = 0$  is 0,

$$|e(x, t, \xi)| \leq (\sin \varphi)^{-\sigma} |t| \langle \xi \rangle^{2d+k} e^{-c \operatorname{Re} t \langle \xi \rangle^d} .$$

This shows (3.35) for  $\alpha = \beta = 0$ . The proof for  $D_x^\beta D_\xi^\alpha e(x, t, \xi)$  is analogous. The estimate (3.36) is obtained similarly to (3.34), using (3.35) instead of (3.33).  $\square$

**Theorem 3.10.** 1° *In local coordinates the kernel terms satisfy for some  $c' > 0$ :*

$$(3.37) \quad |\mathcal{K}_{V_{-d-l}}(x, y, t)| \leq (\sin \varphi)^{-2l} e^{-c' \operatorname{Re} t} \begin{cases} (\sin \varphi)^{(l-n)/d} |t|^{(l-n)/d} & \text{if } d-l > -n, \\ |t| (|\log \operatorname{Re} t| + 1) & \text{if } d-l = -n, \\ |t| & \text{if } d-l < -n. \end{cases}$$

2° *Let  $N = \max\{\frac{n}{d}, \frac{7n}{2} + 4d + 7\}$ , Then the full kernel satisfies*

$$|\mathcal{K}_V(x, y, t)| \leq (\sin \varphi)^{-N} e^{-c' \operatorname{Re} t} |t|^{-n/d} .$$

*For a given  $c_0 > 0$  we can modify  $p^0$  to satisfy  $\inf_{x, \xi} \gamma(p^0(x, \xi)) \geq c_0$ ; then  $c'$  can be any number in  $]0, c_0[$ .*

*Proof.* 1°. For  $l \geq 1$ , the assertion follows from Lemma 3.9, (3.34) resp. (3.36).

On the other hand, we explicitly compute for  $l = 0$

$$|\mathcal{K}_{V_{-d}}(x, y, t)| = (2\pi)^{-n} \left| \int_{\mathbb{R}^n} e^{i(x-y) \cdot \xi} e^{-tp^0(x, \xi)} d\xi \right| \\ \leq e^{-c_1 \operatorname{Re} t} \left( \int_{\mathbb{R}^n} e^{-c_2 \operatorname{Re} t |p_h^0(x, \xi)|} d\xi + \int_{|\xi| \leq 1} (e^{-c_2 \operatorname{Re} t |p^0(x, \xi)|} - e^{-c_2 \operatorname{Re} t |p_h^0(x, \xi)|}) d\xi \right) \\ \leq e^{-c_1 \operatorname{Re} t} \left( \int_{\mathbb{R}^n} e^{-\operatorname{Re} t |\xi|^d} d\xi + 1 \right) \leq e^{-c' \operatorname{Re} t} (\operatorname{Re} t)^{-n/d} .$$

The assertion follows, since  $\operatorname{Re} t \doteq |t| \sin \varphi$ .

2°. We choose  $M = n + 1 + 2d$  in Theorem 3.8 and add  $\mathcal{K}_{V_{-d-l}}$  for  $0 \leq l < M$ . The most singular terms dominate.

See Section 2 for how to obtain the allowed range of  $c'$ .  $\square$

**Theorem 3.11.** 1° In local coordinates,  $\mathcal{K}_{V_{-d}}$  satisfies for some  $c' > 0$ :

$$(3.38) \quad |\mathcal{K}_{V_{-d}}(x, y, t)| \leq (\sin \varphi)^{-n-1} e^{-c' \operatorname{Re} t} |t| |x - y|^{-d-n}.$$

For  $l \geq 1$ , the kernels  $\mathcal{K}_{V_{-d-l}}$  satisfy

$$(3.39) \quad |\mathcal{K}_{V_{-d-l}}(x, y, t)| \leq (\sin \varphi)^{-[n-l+1]_+ - 2l} e^{-c' \operatorname{Re} t} \begin{cases} |t| |x - y|^{l-d-n} & \text{if } d-l > -n, \\ |t| (|\log |x - y|| + 1) & \text{if } d-l = -n, \\ |t| & \text{if } d-l < -n. \end{cases}$$

2° Let  $N = \frac{7n}{2} + 4d + 7$ . Then the full kernel satisfies

$$|\mathcal{K}_V(x, y, t)| \leq \sin(\varphi)^{-N} e^{-c' \operatorname{Re} t} |t| |x - y|^{-d-n}.$$

If  $\gamma(P) > 0$ , we modify  $p^0$  to satisfy  $\inf_{x, \xi} \gamma(p^0(x, \xi)) \geq \gamma(P)$ , then  $c'$  can be any number in  $]0, \gamma(P)[$ .

*Proof.* 1°. For  $l \geq 1$ , we obtain from Lemma 3.9, (3.35), that

$$|D_\xi^\alpha v_{-d-l}(x, t, \xi)| \leq (\sin \varphi)^{-2l-|\alpha|} \langle \xi \rangle^{d-l-|\alpha|} |t| e^{-c' \operatorname{Re} t}.$$

The estimate (3.39) then follows from the kernel estimates in Proposition 2.2.

For  $l = 0$ ,  $v_{-d}(x, t, \xi) = e^{-tp^0(x, \xi)}$ , and we obtain (3.38) as in the proof of Theorem 2.4.

2°. We choose  $M = n + 1 + 2d$  in Theorem 3.8 and add  $\mathcal{K}_{V_{-d-l}}$  for  $0 \leq l < M$ . The most singular terms dominate.  $\square$

The estimates hold also when  $\varphi \in [0, \frac{\pi}{4}[$  is replaced by  $\varphi \in ]-\frac{\pi}{4}, 0]$  and  $\sin \varphi$  is replaced by  $|\sin \varphi|$ . Note that for  $t$  with  $\arg t = \theta \in ]-\frac{\pi}{2}, \frac{\pi}{2}[$ ,  $\cos \theta = |\sin 2\varphi| \doteq |\sin \varphi|$ , when  $\theta = \pm(\frac{\pi}{2} - 2|\varphi|)$  (cf. Remark 3.4). Then we can formulate the final result obtained by combining Theorem 3.10 and Theorem 3.11 as in the proof of Theorem 2.5, as follows:

**Theorem 3.12.** Let  $P$  be selfadjoint strongly elliptic of order  $d > 0$  on  $M$ , with  $\gamma(P) \geq 0$ . The heat kernel  $\mathcal{K}_V$  satisfies for  $t \in \mathbb{C}_+$  with  $\arg t = \theta \in ]-\frac{\pi}{2}, \frac{\pi}{2}[$  the Poisson estimate, where  $N = \max\{\frac{n}{d}, \frac{7n}{2} + 4d + 7\}$ :

$$(3.40) \quad |\mathcal{K}_V(x, y, t)| \leq (\cos \theta)^{-N} e^{-\gamma(P) \operatorname{Re} t} \frac{|t|}{(d(x, y) + |t|^{1/d})^d} ((d(x, y) + |t|^{1/d})^{-n} + 1).$$

*Proof.* In the region where  $|t|^{1/d} \leq d(x, y)$ ,  $d(x, y) \doteq d(x, y) + |t|^{1/d}$ , and the asserted estimate follows from Theorem 3.11. In the region where  $|t|^{1/d} \geq d(x, y)$ , we use  $|t|^{1/d} \doteq d(x, y) + |t|^{1/d}$  and Theorem 3.10.  $\square$

**Remark 3.13.** The  $+7$  in  $N$  partly stems from repeated rounding up to the nearest integer. It may be reduced by choosing  $M + 2d$  closer to  $n$  and using a version of Theorem 3.5 with a non-integer number of derivatives, or sharper versions for negative orders.

Our result applies in particular to the Dirichlet-to-Neumann operator. For this operator ter Elst and Ouhabaz [EO13] have estimates in terms of  $-N$ -th powers of  $\cos \theta$ , where the dimension  $n$  enters nonlinearly in  $N$ .

#### 4. KERNELS OF HEAT SEMIGROUPS FOR PERTURBATIONS OF FRACTIONAL LAPLACIANS AND THE DIRICHLET-TO-NEUMANN OPERATOR

This section complements the general upper bounds from Section 2 with lower estimates in the case of fractional powers of the Laplacian and the Dirichlet-to-Neumann operator.

Let  $\Delta$  be the (nonnegative) Laplace-Beltrami operator on the closed, compact Riemannian  $n$ -dimensional manifold  $M$ ; it defines a selfadjoint nonnegative operator on  $L_2(M)$ , also denoted  $\Delta$ . In this case,  $\Delta^{d/2}$  is an elliptic pseudodifferential operator of order  $d$  on  $M$ , with positive principal symbol  $|\xi|^d$ , defining a selfadjoint nonnegative operator on  $L_2(M)$ ; it generates a holomorphic semigroup  $V^d(t) = e^{-t\Delta^{d/2}}$  with  $C^\infty$ -kernel for  $t > 0$ ,

$$\mathcal{K}_{V^d}(x, y, t) = \langle \delta_x, V^d(t) \delta_y \rangle.$$

The semigroups  $e^{-t\Delta}$  and  $V^d(t)$  are related by subordination formulas. For  $d = 1$ , they assume a simple form:

**Lemma 4.1.** *Let  $\lambda \geq 0$ . One has for  $t \geq 0$ :*

$$(4.1) \quad e^{-t\sqrt{\lambda}} = \frac{1}{2\sqrt{\pi}} \int_0^\infty e^{-s\lambda} t e^{-\frac{t^2}{4s}} s^{-\frac{3}{2}} ds.$$

*Proof.* Let  $\alpha = t\sqrt{\lambda}/2$  and let  $x = \frac{t}{2}s^{-\frac{1}{2}}$ ; then  $dx = -\frac{t}{4}s^{-\frac{3}{2}}ds$ , and equation (4.1) is turned into

$$(4.2) \quad \sqrt{\pi} e^{-2\alpha} = \int_0^\infty e^{-x^2 - \frac{\alpha^2}{x^2}} 2 dx.$$

To show this, note that the left-hand side  $I(\alpha)$  satisfies  $I(\alpha) \in C^1(\mathbb{R}_+)$ ,  $\lim_{\alpha \rightarrow 0+} I(\alpha) = \sqrt{\pi}$ , and for  $\alpha > 0$  (with  $y = \alpha x^{-1}$ ,  $dy = -\alpha x^{-2}dx$ ):

$$\partial_\alpha I(\alpha) = \int_0^\infty e^{-x^2 - \frac{\alpha^2}{x^2}} (-4\alpha) x^{-2} dx = -2 \int_0^\infty e^{-\frac{\alpha^2}{y^2} - y^2} 2 dy = -2I(\alpha).$$

Thus  $I(\alpha) = ce^{-2\alpha}$  with  $c = \sqrt{\pi}$ .  $\square$

By Zolotarev [Z86] (see also Grigor'yan [G03]), there exists for any  $0 < d < 2$  a non-negative function  $\eta_t^d(s)$  such that

$$(4.3) \quad e^{-t\lambda^{d/2}} = \int_0^\infty e^{-s\lambda} \eta_t^d(s) ds.$$

Here  $\eta_t^d$  has the following properties

$$(4.4) \quad \eta_t^d(s) = t^{-2/d} \eta_1^d\left(\frac{s}{t^{2/d}}\right) \quad (s, t > 0),$$

$$(4.5) \quad \eta_t^d(s) \leq ts^{-1-\frac{d}{2}} \quad (s, t > 0),$$

$$(4.6) \quad \eta_t^d(s) \geq ts^{-1-\frac{d}{2}} \quad (s \geq t^{2/d} > 0).$$



By an application of the spectral theorem, we obtain for all  $t > 0$ ,

$$(4.7) \quad V^d(t)f = e^{-t\Delta^{d/2}}f = \int_0^\infty e^{-\tau\Delta}f \eta_t^d(\tau) d\tau, \text{ for all } f \in H^s(M).$$

In view of (4.7), it holds that

$$\langle \delta_x, V^d(t)\delta_y \rangle = \langle \delta_x, \int_0^\infty e^{-\tau\Delta}\delta_y \eta_t^d(\tau) d\tau \rangle = \int_0^\infty \langle \delta_x, e^{-\tau\Delta}\delta_y \rangle \eta_t^d(\tau) d\tau,$$

resulting in an identity for the kernels: For all  $t > 0$ ,

$$(4.8) \quad \mathcal{K}_{V^d}(x, y, t) = \int_0^\infty \mathcal{K}_{e^{-\tau\Delta}}(x, y) \eta_t^d(\tau) d\tau, \text{ for } (x, y) \in M \times M.$$

Using this formula, we can deduce upper and lower estimates for  $\mathcal{K}_{V^d}$  from those known for  $\mathcal{K}_{e^{-\tau\Delta}}$ . The following upper and lower estimates are well-known (see e.g. L. Saloff-Coste [S10]):

$$(4.9) \quad \frac{c_1}{\mathcal{V}(x, \sqrt{\tau})} e^{-C_1 \frac{d(x, y)^2}{\tau}} \leq \mathcal{K}_{e^{-\tau\Delta}}(x, y) \leq \frac{c_2}{\mathcal{V}(x, \sqrt{\tau})} e^{-C_2 \frac{d(x, y)^2}{\tau}}.$$

Here  $\mathcal{V}(x, r)$  denotes the volume of a ball of radius  $r$  around  $x$ . For a closed compact  $n$ -dimensional manifold  $M$ ,  $\mathcal{V}(x, r) \doteq r^n$  for small  $r$ , and  $\mathcal{V}(x, r)$  equals the volume of the connected component containing  $x$  when  $r \geq \text{diam } M$ . Hence

$$(4.10) \quad \mathcal{V}(x, \sqrt{\tau})^{-1} \doteq (\tau^{n/2})^{-1} + 1.$$

**Theorem 4.2.** *Let  $0 < d < 2$ . The kernel of the semigroup  $V^d(t) = e^{-t\Delta^{d/2}}$  satisfies for  $t \geq 0$ :*

$$(4.11) \quad \mathcal{K}_{e^{-t\Delta^{d/2}}}(x, y) \doteq \frac{t}{(d(x, y) + t^{1/d})^d} \left( (d(x, y) + t^{1/d})^{-n} + 1 \right).$$

*Proof.* The upper estimate follows already from Corollary 1.6. The following proof moreover extends to give the lower estimate. Inserting the heat kernel bounds (4.9), (4.10) into (4.8) and using (4.5), we find

$$\begin{aligned} \mathcal{K}_{V^d}(x, y, t) &\leq \int_0^\infty (\tau^{-n/2} + 1) \eta_t^d(\tau) e^{-C \frac{d(x, y)^2}{\tau}} d\tau \\ &\leq t \int_0^\infty \tau^{-n/2} \tau^{-1-\frac{d}{2}} e^{-C \frac{d(x, y)^2}{\tau}} d\tau + t \int_0^\infty \tau^{-1-\frac{d}{2}} e^{-C \frac{d(x, y)^2}{\tau}} d\tau. \end{aligned}$$

By a change of variables  $\tau \mapsto Cd(x, y)^2\tau$ , the first term equals

$$(4.12) \quad t(Cd(x, y)^2)^{-\frac{d+n}{2}} \int_0^\infty \tau^{-\frac{n+d}{2}-1} e^{-1/\tau} d\tau \doteq \frac{t}{d(x, y)^{n+d}}.$$

Similarly, the second term is

$$t(Cd(x, y)^2)^{-\frac{d}{2}} \int_0^\infty \tau^{-\frac{d}{2}-1} e^{-1/\tau} d\tau \doteq \frac{t}{d(x, y)^d},$$

and altogether,

$$\mathcal{K}_{V^d}(x, y, t) \leq \frac{t}{d(x, y)^d} (d(x, y)^{-n} + 1).$$

On the other hand, using the uniform bound  $\mathcal{K}_{e^{-\tau\Delta}}(x, y) \leq \tau^{-n/2} + 1$  and (4.4), we obtain

$$\begin{aligned} \mathcal{K}_{V^d}(x, y, t) &\leq \int_0^\infty (\tau^{-n/2} + 1) \eta_t^d(\tau) d\tau = \int_0^\infty (\tau^{-n/2} + 1) \eta_1^d\left(\frac{\tau}{t^{2/d}}\right) t^{-2/d} d\tau \\ &= \int_0^\infty (t^{-n/d} \tau^{-n/2} + 1) \eta_1^d(\tau) d\tau \doteq t^{-n/d} + 1. \end{aligned}$$

Thus

$$\mathcal{K}_{V^d}(x, y, t) \leq \min\left\{t^{-n/d} + 1, \frac{t}{d(x, y)^d} (d(x, y)^{-n} + 1)\right\}.$$

If  $t^{1/d} \geq d(x, y)$ ,

$$t^{-n/d} \leq t^{-n/d} \left(\frac{d(x, y)}{t^{1/d}} + 1\right)^{-n-d} = t(d(x, y) + t^{1/d})^{-n-d}$$

and

$$1 \leq \left(\frac{d(x, y)}{t^{1/d}} + 1\right)^{-d} = t(d(x, y) + t^{1/d})^{-d}.$$

On the other hand, for  $t^{1/d} \leq d(x, y)$  we have  $d(x, y) \doteq d(x, y) + t^{1/d}$  and hence

$$\frac{t}{d(x, y)^d} (d(x, y)^{-n} + 1) \leq \frac{t}{(d(x, y) + t^{1/d})^d} ((d(x, y) + t^{1/d})^{-n} + 1).$$

This shows “ $\leq$ ” in (4.11).

To show the opposite inequality in (4.11), note that the integrand in (4.8) is non-negative, and (4.9), (4.10) imply

$$\mathcal{K}_{V^d}(x, y, t) = \int_0^\infty \mathcal{K}_{e^{-\tau\Delta}}(x, y) \eta_t^d(\tau) d\tau \geq \int_\alpha^\infty (\tau^{-n/2} + 1) \eta_t^d(\tau) e^{-C \frac{d(x, y)^2}{\tau}} d\tau$$

for  $\alpha = \max\{t^{2/d}, d(x, y)^2\}$ . Now, for  $\tau \geq d(x, y)^2$ ,  $e^{-C \frac{d(x, y)^2}{\tau}} \geq e^{-C}$ . Then by (4.6),

$$\begin{aligned} \mathcal{K}_{V^d}(x, y, t) &\geq \int_\alpha^\infty (\tau^{-n/2} + 1) t\tau^{1-\frac{1}{2}} d\tau \doteq t(\alpha^{-\frac{n+d}{2}} + \alpha^{-\frac{d}{2}}) \\ &= \min\{t^{-n/d}, td(x, y)^{-n-d}\} + \min\{1, td(x, y)^{-d}\} \\ &\geq t(d(x, y) + t^{1/d})^{-n-d} + t(d(x, y) + t^{1/d})^{-d}. \quad \square \end{aligned}$$

For  $d = 1$  this complies well with the explicit kernel formula (2.3) for the Poisson operator solving the Dirichlet problem for the Laplacian on  $\mathbb{R}_+^{n+1}$ .

We also consider the case where  $M$  is the boundary of a compact  $(n+1)$ -dimensional Riemannian manifold  $\widetilde{M}$  with boundary. With  $\Delta$  denoting the nonnegative Laplace-Beltrami operator on  $M$ , we shall compare  $\mathcal{K}_{e^{-t}\sqrt{\Delta}}$  with the kernel of the semigroup generated by the (nonnegative) Dirichlet-to-Neumann operator  $P_{DN}$  on  $M$ .  $P_{DN}$  is the operator mapping  $u$  to the normal derivative  $\partial_\nu \widetilde{u}$ , where  $\widetilde{u}$  is the harmonic function on  $\widetilde{M}$  with boundary value  $u$ . It is known (cf. [G71]) that  $P_{DN}$  is an elliptic pseudodifferential operator of order 1 on  $M$  with the same principal symbol as  $\sqrt{\Delta}$ .

Since  $\Delta^{d/2}$  is a classical strongly elliptic  $\psi$ do of order  $d$ , Theorem 2.5 applies to all operators of the form  $P = \Delta^{d/2} + P'$  with  $P'$  classical of order  $d - 1$ , giving upper estimates of the absolute value of the kernels; note that no selfadjointness is required. For such operators we can also show lower estimates.

**Theorem 4.3.** *Let  $d \in ]0, 2[$  and let  $P$  be a classical  $\psi$ do of order  $d$  with the same principal symbol as  $\Delta^{d/2}$ . Then the kernel of  $V(t) = e^{-tP}$  satisfies for  $t \geq 0$ :*

(4.13)

$|\mathcal{K}_V(x, y, t)| \leq t \left( (d(x, y) + t^{1/d})^{-n-d} + (d(x, y) + t^{1/d})^{-d} \right) + e^{-c_1 t} t (d(x, y) + t^{1/d})^{1-n-d},$   
for any  $c_1 < \gamma(P)$  ( $c_1 = \gamma(P)$  if Corollary 2.6 applies). Moreover, there is an  $r > 0$  such that

$$(4.14) \quad |\mathcal{K}_V(x, y, t)| \geq t (d(x, y) + t^{1/d})^{-d-n}, \text{ for } d(x, y) + t^{1/d} \leq r.$$

*Proof.* As  $P$  and  $\Delta^{d/2}$  have the same principal symbol,

$$V(t) = V^d(t) + V',$$

where  $V'$  is of lower order, more precisely  $V'$  is the difference between the first remainders for  $V(t) = e^{-tP}$  and  $V^d(t) = e^{-t\Delta^{d/2}}$ , as in the second line of (2.25). Hence

$$(4.15) \quad |\mathcal{K}_{V'}(x, y, t)| \leq e^{-c_1 t} t (d(x, y) + t^{1/d})^{1-n-d}.$$

Now (4.11) and (4.15) together imply (4.13).

To obtain the lower estimate (4.14), we note that

$$(4.16) \quad cs^{-n-d} - c's^{1-n-d} = cs^{-n-d}(1 - c'c^{-1}s) \geq 2^{-1}cs^{-n-d}, \text{ when } s \leq c/(2c'),$$

so for  $t$  in a bounded set where  $e^{-c_1 t} \leq c'$ , the lower estimate in (4.11) implies that (4.14) holds for small  $d(x, y) + t^{1/d}$ .  $\square$

We can also obtain upper and lower estimates for the Dirichlet-to-Neumann operator.

**Theorem 4.4.** *The kernel of  $e^{-tP_{DN}}$  satisfies for  $t \geq 0$ :*

$$(4.17) \quad \mathcal{K}_{e^{-tP_{DN}}}(x, y, t) \leq \frac{t}{d(x, y) + t} \left( (d(x, y) + t)^{-n} + 1 \right),$$

and there is an  $r > 0$  such that it satisfies

$$(4.18) \quad \mathcal{K}_{e^{-tP_{DN}}}(x, y, t) \geq t (d(x, y) + t)^{-1-n}, \text{ for } d(x, y) + t \leq r.$$

*Proof.* Here  $P_{DN}$  is known to be selfadjoint nonnegative, and the semigroup has real, nonnegative kernel ([AM07], [AM12]), so that we may omit absolute values. The upper estimate (4.17) follows from Corollary 2.6. The lower estimate (4.18) follows from Theorem 4.3 since  $P_{DN}$  differs from  $\Delta^{1/2}$  by a classical  $\psi$ do of order 0.  $\square$

**Remark 4.5.** This work was inspired from a conversation of the second author with W. Arendt and A. ter Elst in August 2012, where we suggested the applicability of pseudo-differential methods as in [G96] to the Dirichlet-to-Neumann semigroup. We have very recently learned of the efforts of ter Elst and Ouhabaz in [EO13], giving an analysis of the Dirichlet-to-Neumann semigroup by somewhat different methods, and obtaining some of the same results as those presented here.

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